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Tables of Trigonometric Functions in Non-Sexagesimal Arguments

Excluding the ordinary tables of trigonometric functions in sexagesimal arguments the two principal groups of such tables are those with arguments in A. *Radians*,—tables of this type have been already listed in RMT **81**; and B. *Grades*. But we shall also consider tables of the trigonometric functions with arguments in C. *Mils;* D1. *Gones;* D2. *Cirs;* and E. *Time*.

B. *Grades*. The first stage in the evolution of the centesimal division of the quadrant was by the division of each sexagesimal degree into 100 minutes and each of these minutes into 100 seconds. This was conceived already about 1450 by Theodericus Ruffi, in a Latin codex in Munich.[1] The centesimal division of the degree was also presented by Vieta in his *Calendarij Gregoriani*, 1600, p. "29"(*Opera Mathematica*, 1646, p. 487). It was this source which suggested to Henry Briggs the choice of every hundredth of a degree as the unit in the preparation of his wonderful "sine canon";[2] see RMT **79**.

Late in the eighteenth century the centesimal division of the quadrant was definitely in the thought of scholars then in Berlin and especially of JOHANN CARL SCHULZE (1749–1790), author of *Neue und erweiterte Sammlung logarithmischer, trigonometrischer und anderer zum Gebrauch der Mathematik unentbehrlicher Tafeln* [also French t.p.], Berlin, 2v., 1778. He had studied under J. H. Lambert (1728–1778) and became a member of the Academy of Sciences at Berlin, in which Lagrange (following Euler) was director of its mathematical section for a score of years before he moved to Paris in 1787. In the preface of the first volume of his *Tafeln*, and also in his work of 1783 (referred to by Mehmke in his splendidly documented report[3]) Schulze appears to have made ready for the press a manuscript of a seven-place table of natural trigonometric functions and of their logarithms for every thousandth of a sexagesimal degree. Since Lagrange expressed the opinion that a table for arguments involving the centesimal division of the quadrant (into grades) would be more desirable, Schulze gave up the plan of publishing his table and promoted the calculation of the type suggested by Lagrange. A substantial table of this type did finally appear in Berlin, nearly a decade after Schulze's death. This was

(1). JOHANN PHILIPP HOBERT (1759–1826) and CHRISTIAN LUDWIG IDELER (1766–1846), *Neue trigonometrische Tafeln für die Decimaleintheilung des Quadranten*, Berlin, 1799, lxxii, 353 p. 11.7×20.5 cm.

On p. 1–309 is a table of sines, cosines, tangents, cotangents, and of their logarithms for every $10^{``}$ to $3^{g}$, and then for each $1^{`}$ to $50^{g}$, to 7D, with differences. On p. 313–315 are tables to 10D, with differences, of natural sines and tangents for $0^{``}(1^{``})1^{`}(1^{`})1^{g}$. Of this volume Delambre has well written, "Cet ouvrage m'a paru d'une correction et d'une exactitude rares." In (2). *Astronomisches Jahrbuch für das Jahr 1798*, Berlin, 1795, J. E. BODE published an unimportant table for these same 8 functions, for each half grade ($=27'$) to 7D. An earlier table in the centesimal division served Hobert and Ideler as a valuable check on their computation. But before considering this we shall refer to another matter which contributed to the persistence of centesimal tables to the present day.

I refer to the movement which culminated during the decade of the French revolution in the establishment of the Metric System. In 1799 a law was passed definitely fixing the unit of length, a *meter*, as one ten millionth of the quadrant of the terrestrial meridian through Paris; the unit of volume, a *liter*, as equivalent to a cube of edge one tenth of a meter; the unit of weight, a *gram*, as equivalent to a cubic centimeter of distilled water at the maximum density, 4°C, weighed *in vacuo*. This new system became compulsory in 1801. Delambre, Lagrange, Laplace, Legendre and Monge were mathematicians among members of the Commission establishing these units. In 1920 the Metric System was the only legal one in 35 countries, and its use was also allowed in 11 others, including Great Britain and United States.

On representations of the mathematician L. N. M. Carnot, and others, the French Government decided in 1784 that new tables of the sines, tangents, etc., and their logarithms, should be constructed. The execution of this task was in 1792 put in the hands of GASPARD CLAIR FRANÇOIS MARIE RICHE DE PRONY (1755–1839), then director of the Bureau du Cadastre. Hence the resulting tables are referred to as (3). *Tables du Cadastre.* In order to call special attention to the new French Metric System Riche de Prony was expressly charged "non seulement à composer des tables qui ne laissassent rien à desirer quant à l'exactitude, mais à en faire le monument de calcul le plus vaste et le plus imposant qui eût jamais été exécuté ou même conçu." To carry through the work almost unlimited powers were given to the director in the choice and organization of his collaborators and computers.

These were divided into three groups; the first, consisting of four or five mathematicians (with Legendre as president), was occupied with analytic work and the calculation of fundamental numbers. The second group contained seven or eight skilled computers, working on formulae supplied by the first group, and checking work by the 70 or 80 members of the third group endowed with no great mathematical abilities. In fact they were mainly recruited from among hairdressers whom the abandonment of the wig and powdered hair in men's fashions, had deprived of a livelihood. The work on the tables, which was done wholly in duplicate, by two divisions of the computers, was finished in two or three years. Until recently, at least, one set of the 17 large folio volumes of ms. was in the library of the Paris Observatory, and the other set in the Library of the Institut de France. In each set 8 volumes were devoted to logarithms of numbers to 200,000. For this article it is only the contents of the remaining 9 volumes which are of interest.

There was one volume of natural sines for each centesimal minute in the quadrant, to 25D, with 7 or 8 columns of differences; to be published, to 22D, with 5 columns of differences. Four volumes were devoted to: (a) log sin a throughout the quadrant for each tenth of a centesimal minute, to 14D, with 5 columns of differences; (b) log sina/a for $a=[0^g\!.00000(0^g\!.00001)0^g\!.05000;$ 14D], with 5 columns of differences.
And finally, there were four volumes for (a) log tan a, and (b) log tan a/a,for the same ranges as for log sin a, and log sin a/a.

The mss. of the Institut have also a volume of the first 500 multiples of certain sines and cosines. In the more complete mss. of the Observatory two v. in addition to the 17v., include a 63-page Introduction in Riche de Prony's handwriting, and such subsidiary tables as, (a) the first 26 powers of $\pi/2$, to

28S; (b) log sin a, to 14D, with 8 orders of differences, for varying values of a throughout the quadrant.

An arrangement was made for the publication of a stereotype folio volume of these tables, to contain 1200 p. exclusive of the introduction. The one hundred plates made seem to have included all the table of natural sines; but financial difficulties prevented the continuation of the undertaking. De Morgan tells us that "a distinguished member of the Board of Longitude, London, was instructed by our Government to propose to the Board of Longitude of Paris to print an abridgement of these tables, at the joint expense of the two countries. £5000 was named as the sum which our Government was willing to advance for this purpose; but the proposal was declined." Note the French statement in *Nouv. Ann.*[4] "Il y a eu des négociations à ce sujet avec le gouvernement anglais qui n'aboutirent pas."

The first printed table for the centesimal division of the quadrant was in that remarkable collection of tables of

(4). JEAN FRANCOIS CALLET (1744–1799), *Tables Portatives de Logarithmes*, first stereotyped ed., Paris, 1795. 14×22.2 cm. There have been many editions, even down to 1906.

In this large volume there is one table (50 p.) containing the logarithms of the sine, cosine and tangent, for [$0^g(1^`)50^g$; 7D], with differences. This is followed by a table (10 p.) of the natural sines and cosines, to 15D, and of their logarithms, to 14D, for every $10^`$. For the first of these tables

(1). Hobert and Ideler list (*l.c.*, p. 348–349) 375 seventh-place unit errors, and one other error. (5). Borda and Delambre list (p. 120) one error in this table and four in the second; a fifth listed error was not an error in the Brown University copy. All of these errors were already corrected in the (4A). 1821 tirage of Callet's tables.

We come next to an excellent volume issued from "L'Imprimerie de la République."

(5). *Tables Trigonométriques Décimales ou Table des Logarithmes des Sinus, Sécants et Tangentes suivant la Division du Quart de Cercle en 100 Degrés, du Degré en 100 Minutes, et de la Minute en 100 Secondes; précédées de la Table des Logarithmes des Nombres . . . et de Plusieurs Tables Subsidiaires, Calculées* par JEAN CHARLES BORDA (1733–1799), *revues, augmentées et publiées* par J. B. J. DELAMBRE. Paris, 1801. 17.5×22.3 cm.

The table in which we are interested is a seven-place logarithm table of all six of the trigonometric functions from 0^g to 3^g for every $10^{``}$ (with full proportional parts for every second) and thence for every $1^`$, with full proportional parts for every $10^{``}$. Delambre tells us (p. 39) that these were the first tables for the centesimal division of the quadrant that were ever made, since Borda had already finished the manuscript in 1792. He tells us also that Callet (1795) used Borda's manuscript as well as the *Tables du Cadastre* in preparing his tables. Since Hobert and Ideler had also used Callet we thus have three tables materially indebted to the *Tables du Cadastres*, with which Delambre especially made a detailed comparison of parts common with Borda's manuscript. The tables of Hobert and Ideler, and Borda, are in argument of exactly the same extent, but the former does not give secants nor proportional parts all calculated; on the other hand it does contain values of the natural sines

and tangents. Delambre, no. (5), states these facts exactly, in contrast to Glaisher whose remark might readily mislead.[5]

We shall next refer to important (6). Manuscripts of EDWARD SANG (1805–1890), author of the notable *A New Table of Seven-Place Logarithms of all Numbers from* 20 000 *to* 200 000, London, 1871. Of these mss. the Royal Society of Edinburgh is now custodian for the British Nation.[6] From Henderson I learn that of the mss. in which we are now interested the first was a Canon of Sines to 33D for each two-thousandth part of the quadrant (finished in 1877). There followed (1881) a Canon of Sines to 15D for each centesimal minute, with first and second interscript differences; and (1888) Logarithmic Sines and Tangents to 15D for each ten-thousandth of the quadrant.

The following quotation from the abstract written by Sang (Henderson) is not without interest:

For the utilisation of a method of solving Kepler's problem the values of circular segments measured in degrees of surface for each of the 40 000 minutes of the circumferences and tables of the true anomalies for each degree in orbits of each degree of ellipticity were computed and laid on the table of the Royal Society of Edinburgh in July 1879. For mean anomalies, etc., it became necessary to calculate a canon of the centesimal division. In Callet's *Tables Portatives* there is a table to 7D for every minute and on examination on the page for 31^g there were 23 errors of discrepancy in the differences."

We have already noted that the great (3). *Tables du Cadastre*, were never published. In 1888, however, the French Minister of War requested the Service Géographique de l'Armée to prepare two tables, a large eight-place table, to be abridged from Riche de Prony's ms., and a five- (four-) place table for more general use. The first work, prepared by Mm. VILLEDEUIL and L'HÔPITAL, has the following title:

(7). *Tables des Logarithmes à huit Décimales des Nombres entiers de* 1 *à* 120 000 *et des Sinus et Tangentes de dix Secondes en dix Secondes d'Arc dans le Système de la Division Centésimale du Quadrant*, Paris, Imprimerie Nationale, 1891 [viii, 622 p.]. 27.5×34.7 cm. Preface signed by Général Derrécagaix, director of the Service. The large unnumbered pages 225–622 are devoted to a table of logarithms of sines, cosines, tangents, cotangents, to 8D, for every 10'' (semiquadrantal), with p.p. for each second. S and T are given for every centesimal minute up to 5^g.

The second (though first published) work prepared by the Service Géographique de l'Armée had the following title:

(8). *Nouvelles Tables de Logarithmes à cinq Décimals pour les Lignes Trigonométriques dans les deux Systèmes de la Division Centésimale et de la Division sexagésimale du Quadrant, et pour les Nombres de* 1 *à* 12 000 *suivies des mêmes Tables à quatre Décimales et des diverses Tables et formules usuelles.* Paris, Imprimerie Nationale, 1889, 235 p. 17.3×25.5 cm. Preface by Colonel Derrécagaix, director of the Service Géographiques de l'Armée. On pages 65–164 are given the logarithms of sines, cosecants, tangents, cotangents, secants, and cosines, to 5D, for each 1', with differences and p.p. Then follow tables, to 5D, of S and T, for every 2' in the first 3^g. On p. 212–215 are logarithms of sines, cosines, tangents, cotangents, to 4D, for every 10' in the quadrant; on p. 220–223 is a similar table for the natural functions.

After 1904, students at the École Polytechnique and École Saint-Cyr were

required to be able to use tables of trigonometric functions with centesimal divisions of the arc. For these schools was published a special edition

(8A). *Nouvelles Tables* . . . [as above] *pour les Nombres* 1 *a* 12 000. *Edition Spéciales a l'usage des Candidats aux Écoles Polytechnique et de Saint-Cyr,* Paris, 1914, reprinted on much better paper 1924, same size page as previous edition, but with the last 30 p. omitted. The preface of each print is signed by Bourgeois, and dated "Paris, 1917."

According to Henderson a second edition of the 1889 work was

(8B). o *Nouvelles Tables* [as in the first edition]. *Deuxième édition revue et corrigée,* Paris, 1901, with a table of errata in the first edition; preface by Général Bassot, director of the Service Géographique. I have seen

(8C). *Nouvelles Tables* [as in the first edition]. *Deuxième édition revue et corrigée,* Paris, 1914. The preface to this second edition is dated Paris, 1914 and is signed by Général Bourgeois, director of the Service Géographique. On p. 236 there is "Errata à la première édition." These were corrected in no. (8A) listed above. And finally, there is

(8D). *Nouvelles Tables* [as above], Paris, 1927. The errata list has disappeared and p. 236 is blank.

We shall presently list a third work prepared by the Service Géographique de l'Armée.

In Germany during the decade 1885–1895 those dealing with problems of surveying and geodesy felt the great need of a table of logarithms of the trigonometric functions for centesimal divisions of the quadrant, since those of Callet, Hobert and Ideler, and Borda and Delambre, were not easily obtainable. At this juncture appeared at Paris the great 8-place table of the Service Géographique de l'Armée, which would also serve the needs of those desiring a seven, six, or five-place table. Nevertheless WILHELM JORDAN (1842–1899), who was the author of the remarkable three-volume work *Handbuch der Vermessungskunde,* decided that he would be rendering a service to science by computing a new table,

(9) *Logarithmisch-Trigonometrische Tafeln für Neue (Centesimale) Theilung mit sechs Decimalstellen.* Stuttgart, 1894, viii, 420 p. 18.5×26.5 cm.

These tables contain the logarithms of the sine, tangent, cotangent, and cosine (p. 155–414), for $[0^g(10^{\prime\prime})20^g.10; 20^g(1^\prime)50^g; 6D]$, and are arranged semiquadrantally. S and T values are given for $[0^g(1^\prime)2^g.50; 6D]$. On p. 416 are newly calculated log sin x and log cos x, for $x=[0^g(1^g)50^g; 15D]$. Between these key values others were interpolated.

The next three posthumous editions (9*A*). o second (1915), (9*B*). o third (1921), and (9C). fourth (1931), were edited by H. P. O. Eggert. Corrections of errors were made in publishing the second and third editions, but no new errors were found to correct for the fourth.

We shall next refer to the great fundamental tables of the trigonometric functions, both logarithmic and natural, by MARIE HENRI ANDOYER (1862–1929). These will probably be the basis, direct or indirect, of all other similar tables of value for a century to come. The first is entitled

(10). *Nouvelles Tables Trigonométriques Fondamentales (Logarithmes),* Paris, Hermann, 1911, 21.8×29 cm.

Table III (p. 9–15) gives the logarithms of sines, cosines, and tangents for each grade to 50^g, to 17D, with variations of various orders, Table IV (p. 17–37)

may be regarded as a table of the logarithms of sine, cosine, and tangent, for $0^g(1/6^g)50^g$, to 5D, and other material. The long introduction (p. ix–xxxii) explains the basis of the computation coupled with criticisms of earlier original work, including that of Jordan. The second volume of Andoyer which is also here of interest to us is entitled.

(11). *Nouvelles Tables Trigonométriques Fondamentales* (*Valeurs Naturelles*), v. 1, Paris, Hermann, 1915, 24×31.3 cm.

Table II (p. 5–21) contains the values of all six of the natural circular functions and of $g(y)=(200/\pi y)-\cot y^g$ and $h(y)=\csc y^g-(200/\pi y)$, for $y=0^g(1^g)50^g$, to 20D; also variations of different orders for the sine, cosine, tangent and secant, and for the functions g and h. Table III (p. 23–65) may be regarded as giving the values of all six of the natural trigonometric functions for $0^g(1/6^g)50^g$, to 17D, and other material.

Turning next to a table which was prepared primarily for use with the Brunsviga calculating machine we have

(12). G. Steinbrenner, *Fünfstellige Trigonometrische Tafeln neuer Teilung* (*Dezimalteilung des Quadranten*) *zum Maschinenrechnen nebst einer zehnstelligen Hilfstafel und einer goniom.—trigonom. Formelsammlung*. Brunswick, 1914. 174 p. 17.2×25 cm.

E. Reich, *Anhang zu G. Steinbrenner: Fünfstellige Tafeln. Hilfstafel für Berechnung der trigonometrischen Richtungskoeffizienten*, Brunswick, 1914, 11 p.

Table I (p. 5–115) gives the values of the natural sines, cosines, tangents, cotangents, for every $1'$ of the quadrant, to 5D, as well as cotangents for $[0^g(1'')0^g15'(10'')3^g$; 5D], with p.p.

Table V (p. 134–139) has (a) sines and cosines for each $10'$ of the quadrant to 10D; (b) sines, cosines, and tangents for $[0'(10'')10'$; 10D], all with p.p.

Table VII (p. 142–150) has cotangents for $[0'(1'')10'(10'')1^g10'$; 10D], and sines and tangents for $[0^g(1')1^g$; 10D].

We may next refer to a work published in China and written by an engineer on its national railway,

(13). E. A. Slosse, *Tables des Valeurs Naturelles des Expressions Trigonométriques. Division Centésimale. Suivies de Tables relatives au tracé de courbes et au calcul des levés trachéométriques*, Tientsin, China, Imprimerie de la Mission du Tcheli, S. E., s.d. xxxvi, 425 p. Preface signed "E.A.S., Chengchow, 1923". 12.6×17.8 cm.

The main table I (p. 1–401) gives the values of all of the natural trigonometric functions, the versed sine, and versed cosine, for each half minute, to 6D, for 200 grades. Table III (p. 415–424) contains the values of $\sin^2 v$ (haversine$2v$), and $\cos^2 v$, for $v=[50^g(1')150^g$; 4D]. Illustrations show how these functions arise in surveying. We now come to

(14). International Geodetic and Geophysical Union, Association of Geodesy, Special Publication, no. 1, 8 *Place Tables of the Natural Values of Sines, Cosines and Tangents according to the Centesimal System, for each Centigrade from* 0 *to* 100 *grades, computed under the direction of* M. Roussilhe . . . , by M. Brandicourt, . . . *followed by* 20 *place tables of the Natural Values of the six trigonometrical functions according to the centesimal system, for each grade from* 0 *to* 100 *grades, taken from the tables* of M. Andoyer. Paris, 1925, [144 p.]. 18.5×16.8 cm. There is also a French title page and introduction (p. 1–12).

This 8-place table, with differences, for each centesimal minute, occupies

p. 25–125, which supplements no.(7), with 8-place logarithms of the circular functions. In spite of the title the tangents 50^g–100^g can not be read directly from the table. The Anodyer items is taken completely from his work of 1915; see Table II of no. (11) above.

The publication of these tables was first suggested at the International Geodetic and Geophysical Union held at Rome in 1922. At first the tables were to give 7 decimals, but in a conference of 1924 at Madrid, it was finally resolved to publish 8 decimal. With such tables it was estimated that an accuracy of 1 millimeter in 50 kilometers would be obtained.

There are the following errors corrected by hand in the Brown University copy of this edition:

In D-column for sinus 7^g bet. 02 and 03, *for* 17612 *read* 15612.
In D-column for cosinus 7^g bet. 06 and 07, *for* 1339 *read* 1739.
In tan $14^g.65$, *for* 0.2342 6170 *read* 0.2342 7170.
Other places where the type is blurred and a number can not be read are: tan $2^g.66=0.0418\,0751$; cos $30^g.50=0.8874\,1345$; sin $43^g.15=0.6270\,7976$; D, after cos $47^g.34$, 10634.

These are corrected in the

(14A). "New revised edition" (Paris, 1933) which has the heading "International Geodetic and Geophysical Union Association of Geodesy." The number of pages is the same. The table of sines is based on the 15-place table of Callet, no. (5), and the table of tangents on the 15-place table of Andoyer, 1916.

Another work of importance is that of

(15). Johann Theodor Peters (1869–), *Sechstellige trigonometrische Tafel für jede Minute des in hundert Grade geteilten Quadranten.* Berlin, Gebr. Wichmann, 1930. vi, 170 p. 17×25.8 cm.

Table I gives (p. 1–44) the values of cosecant and cotangent for $0^g(10'')10^g$, to 5 or 6 S, with p.p.; also the values of $w^g\cdot\cot w$ and $w^g\cdot\csc w$ for $w=0^g\cdot000-2^g\cdot000$; whence, for example, $\csc w = w \csc w/w$. In Table II (p. 45–145) the values of all of the circular functions are given for every centesimal minute, with p.p.

In October 1938 Mr. Peters reported to Mr. Comrie that he had (16). seven and (17). eight-place tables of the trigonometric functions for every 0.001^g, and a (18). ten-place table for every 0.01^g. Thus tables with centesimal division[7] of the quadrant are regarded as of importance even in recent times in Germany, especially for dealing with problems of surveying and ballistics.

A great work of science in which the centesimal division of the quadrant and day is used, is Laplace's *Mécanique Céléste* (5 v., Paris, 1799–1825).

C. *Mils.* The third work prepared by the French Service Géographique de l'Armée was entitled

(19). *Tables de Logarithmes à cinq Décimales pour les Nombres de* 1 *à* 12 000 *et pour les Lignes Trigonométriques dans le Système de la Division de la Circonférence en* 64 000 *parties égales* (*dixième du millième de l'Artillerie*). Paris, Imprimerie Nationale, 1916, [232 p.]. 17×25.7 cm.

The first 64 pages of logarithms of numbers are the same as in the other five-place tables of the Service Géographique. Then follows a five-place table, with p.p., of the logarithms of each of the six trigonometric functions for every sixty-four thousandth of the circumference. $2\pi(10{,}000)\approx62831.8$, which may be rounded of to 64 000. I am unfamiliar with any similar extensive printed table

of this "artillery unit" or mil, 1/16000 of a quadrant, or 20.25"; $1^g = 160$ mils. On the first of the 80 double pages of the table we find the values for the ranges 0(1)100, 15900(1)16100, 31900(1)32100, 47900(1)48100, 639000−64000. High accuracy for this peculiar table may be assumed since it was prepared by Henri Andoyer. It supplied needs of the French artillery units in the last great world war. On p. 229 there is a table of log(sin x/x), and of log(tan x/x), for $x = 0.0000(.0004)0.0600$.

On punched cards of the International Business Machines Corporation are many sets of cards of mathematical tables. Among the sets at PRINCETON UNIVERSITY is one of (20). 8000 cards of a table of the 6 natural trigonometric functions as well as of ½ tangent and ½ cotangent, for every tenth of a mil, to 5D, throughout the quadrant. About a year ago this table was being expanded to include the logarithms of the sine, cosine, and tangent, for the same range.

A reference may be given to R. S. Burington, "The mil as an angular unit and its importance to the army," *Amer. Math. Mo.*, v. 48, 1941, p. 188–189.

D1. *Gones*. The type of trigonometric tables here to be considered is chiefly represented by a large work of the Mexican engineer and geographer, Joaquin de Mendizábal Tamborrel (1856–1926). In his choice of the circumferences as the unit (a *gone*) he followed Villarceau,[8] and divided this into a million parts. The terms decigone, centigone, miligone, . . . microgone were applied to angles 1/10, 1/100, 1/1000, · · · , 1/1000000 of a gone; thus 1 microgone $= 1''.296$. 1 gone $= 100$ cir $= 1000000$ microgones $= 400^g = 360°$. The table in question is

(21). J. DE MENDIZÁBAL TAMBORREL, *Tables des Logarithmes à huit Décimales des Nombres de* 1 *à* 125 000 *et des fonctions goniométriques sinus tangente, cosinus et cotangente de centimiligone en centimiligone et de microgone en microgone pour les 25000 premiers microgones et avec sept décimales pour tous les autres microgones*. Paris, 1891. [ix, 307 p.], 26.7×36.5 cm.

At the foot of pages 1–59 of logarithms of numbers, are given the values of S and T for the first 12500 microgones, *i.e.*, to 4°30′, to 8D. Beginning with page 60 each page corresponds to a miligone; the logarithms of sines and tangents are on the left-hand pages and the logarithms of cosines and cotangents, for the same range, on the opposite right-hand pages. The values are given in (a) a single entry table for [0(10)125000; 8D]; (b) a double-entry table for every microgone, 0 to 25000, to 8D; 25000 to 125000, to 7D. P.p. are given from 12000 on. Suitable headings at the top and bottom of each page lead to results for any number of microgones in the circumference. These remarkable tables were mainly the result of original 10-place calculations completed in 1888. Only the logarithms of the numbers 1–1200 were taken from Callet's Table and those of the numbers 100,000 to 108,000 from Schrön's. (The statement of the *Fortschritte* review in this connection is incorrect.) They were checked against the manuscript (3). *Tables du Cadastre* and the printed volume (1891) of the Service Géographique de l'Armée. The following table published by the same author a few years later is hardly worth a reference:

(22). "Tables numériques d'après la division décimale de la circonférence et du jour," Sociedad Científica "Antonio Alzate," *Memorias*, Mexico, v. 13, nos. 7–8, 1903, math. suppl., 12 p. 16×21.7 cm.

Table II (p. 4–9) gives the values of the circular and hyperbolic functions (sin or tanh, csc or coth, tan or sinh, cot or csch, sec or cosh, cos or sech) to 4D, for the range 0(.25)250, each unit being 1000 microgones. Table III (p.

9–12) contains the logarithms of the same functions for the same range. Compare RMT **89**, Vassal.

D2. *Cirs.*—We have noted (p. 13) that (23). BRIGGS in his *Trigonometria Britannica* gave a table (p. 43) of sines to 19D for all angles obtained by dividing the quadrant into 144 equal parts. It is not appropriate here to discuss why or how such a table illustrates his discussion, and the same remark applies to what now follows. For fuller information the curious reader may either turn to the work itself or to the detailed description by Delambre.[9] Briggs remarks that some zealous amateurs have desired a new division of the circle, which is centisimal, so that the quadrant is divided into 25 units, or cirs, or centigones. To facilitate the preparation of a table of natural sines for thousandths of such a unit, he gives a curious table to 16D for each 0.625 of a unit in the quadrant, the first four and last four of the 40 entries being as follows:

0.625	2°15′	.03925 98157 59068 6
1.250	4 30	.07845 90957 27845 1
1.875	6 45	.11753 73974 57837 7
2.500	9 00	.15643 44650 40230 8
. . .	. . .	. . . .
23.125	83 15	.99306 84569 54926 3
23.750	85 30	.99691 73337 33128 0
24.375	87 45	.99922 90362 40722 9
25.000	90 00	1.00000 00000 00000 0

Briggs notes that this table may then be extended to each 0.125 of a unit, and then to each 0.025, each 0.005, and to each 0.001. The methods of Briggs were also used by Sang (see R. So. Edinb., *Proc.*, v. 9, p. 345–347).

A later table with arguments in this unit is the quite unimportant one of

(24). JOSEPH CHARLES FRANÇOIS REY-PAILHADE (1850–), "Table des logarithmes à quatres décimals de toutes les lignes trigonométriques dans la division décimale du cercle entier," Société Géographique de Toulouse, *Bull.*, v. 19, 1900, p. 94–103. 13.5×21.4 cm. Also issued as a pamphlet, Paris, 1900, 12 p.

To this author is due the name Cir (1/100 of the circumference = 10 miligones). The table is for each of the circular functions for every tenth of a cir in the quadrant; and also for log(sin x/x) and log(tan x/x) for $x=0.0(0.1)2.5$. Since 100 cirs and a gone are identical, all of the entries (except those for log cosec and log sec) are given much more extensively in no. (21).

E. *Time.* In astronomy right ascension, hour angle, and other related quantities, are measured in hours, minutes, and seconds of time. To use with that, tables of trigonometric functions in sexagesimal arguments, called for the employment of conversion relations (such as $6^h=90°$, $1^h=15°$, $4^m=1°$, $1^m=15'$, $1^s=15''$), a source of frequent error. Hence a number of tables of circular functions with time as argument have been compiled or computed during the past century. We list a few of the logarithmic tables, 5D–7D, and tables of natural functions, 5D–10D. The poor first table came from an observatory, namely:

(25). GEORGE BIDDELL AIRY (1801–1892), *Appendix to the Greenwich Observations, 1837: containing no. III, Logarithms of Sines and Cosines to Time, for every ten Seconds through the twenty four hours. Computed under the direction . . . , printed first in 1838: reprinted*, London, 1864, xliv p. 24.5×31.7 cm.

Two large folio pages are devoted to each of the 24 hours of the day; on one page is log sin for each 10^s of 60^m, to 5D, and on the opposite page is log cos. It is thus an "eight-fold repetition of one table" (De Morgan). The phrase

"computed under the direction" is applied to a mere rearrangement of some five-place table. We may next note the title of

(26). Robert Shortrede (1800–1868), *Logarithmic Tables, to Seven Places of Decimals containing . . . Logarithmic Sines and Tangents to every second of the Circle with Arguments in Space and Time.* Edinburgh, 1844. iv, 597 p. 17.3×24.5 cm. (26A). Second ed., 1849; (26B). another ed., 1848; (26C). posthumous and rev. ed. London, 1873.

This notable table was the most complete second canon in existence at the time of its publication. Beside every sexagesimal second of argument is given the corresponding time argument. Of two earlier second canons was the posthumous work of Michael Taylor (1756–1789) *Tables of Logarithms of all Numbers, from* 1 *to* 101 000 *and of the Sines and Tangents to every Second of the Quadrant.* London, 1792. This work and Shortrede's were the basis of

(27). Norbert Herz (1858–), *Siebenstellige Logarithmen der trigonometrischen Functionen für jede Zeitsecunde, zum astronomischen Gebrauche herausgegeben.* Leipzig, 1885. iv, 182 p. 16.4×24.1 cm. Each page is devoted to a minute of time and the logarithms of sines, tangents, cotangents, cosines, to 6D, with differences and p.p. S and T are given for 0^h0^m to 0^h20^m. In the preface are noted two errors in Taylor's table, and one in Shortrede's, not previously listed. We turn back chronologically to

(28). John Caulfield Hannyngton (1807–1885), *Haversines Natural and Logarithmic used in Computing Lunar Distances for the Nautical Almanac,* London, 1876. 327 p. 20.2×32.1 cm.

Versine $t=1-\cos t=2\ \sin^2(t/2)$; haversine $t=\sin^2(t/2)$. The first table (p. 2–145) is for twelve hours and gives log haversine for every time second, 5^m on each page, to 5D, with differences and p.p. Corresponding to each time second, the angular argument is also given. In the table of natural haversines (p. 148–327) the argument is for each $10''$ (with corresponding time arguments) from 0° to 180° (12 hours), to 7D, with p.p. An admirably printed volume. In chronological order we may next list

(29). Friedrich Bidschof (1864–1915) and Arthur Vital, *Fünfstellige mathematische und astronomische Tafeln. Zum Gebrauche für Mathematiker, Astronomen, Geographen, und Seeleute zusammengestellt und mit Formelsammlung versehen.* Stereotype-Ausgabe. Vienna and Leipzig, 1905, xviii, 219 p. 16.7×25.8 cm.

Table 8 (p. 58–115) is a table of the 6 circular functions for every $1'$, to 5D, with differences and with corresponding time arguments, each 4^s. Table 11 (p. 104–115) has log haversine t for $t=[0^h(5^s)12^h; 5S]$. There are many other navigation tables with time as argument, but reference will be given to only one, which has passed through many editions, 1830–1910, namely:

(30). James Inman (1776–1859), *Nautical Tables designed for the Use of British Seamen,* new ed., London, 1858.

The logarithms of all six of the trigonometric functions, as well as of haversines, are given for every second of time, to 6D, p.[37]–[216], and 217–275. And now one more volume in the remarkable series by

(31). J. T. Peters, *Fünfstellige Logarithmentafel der trigonometrischen Funktionen für jede Zeitsekunde des Quadranten,* Berlin, 1912. iv, 83 p. 18.8 ×26.6 cm.

This is simply a table of the logarithms of sines, tangents, cotangents, cosines, for each time second, to 5D. On each page are the values for 2^m, with p.p.

throughout the table, but no differences. In another table we find log sin and log tan for 0^h0^m to 0^h8^m, and of log cos and log cot for 5^h52^m to 6^h0^m, for each tenth of a time second. At least as long ago as 1935 Peters had also prepared (32) a six-place table of the natural trigonometric functions for every second of time.

Further tables of natural trigonometric functions are those computed by

(33). [**D, Q**].—W. Dieckvoss and H. Kox, *Sammlung von Hilfstafeln der Hamburger Sternwarte in Bergedorf. J. Tang* 0^h0^m—*Tang* 1^h0^m, and *K.* cos 0^h0^m—cos 1^h0^m, 1935. 64 p. and 64 p. 19.2×26.3 cm.

Table J is for [$0^h0^m(0.1^s)1^h0^m$; 7D], with p.p., and Table K for the same range and extent. These were the first tables of the natural functions to 7D (if one excludes Hannyngton's Haversine table), and were published for the specific purpose of astrographic plate reductions.

We began this section by listing a very unworthy publication, no. (25), of the Greenwich Observatory, but approach the section's conclusion with an account of a high-class work of importance prepared by H. M. Nautical Almanac Office. This is

(34). [**D, Q**].—[L. J. Comrie], *Seven-Figure Trigonometrical Tables for every Second of Time*. London, His Majesty's Stationery Office, 1939. 101 p. 17.3×26.2 cm.

In December 1938 the Astronomer Royal wrote as follows: "The preparation of these tables was commenced and completed under the direction of Dr. L. J. Comrie, late Superintendent, about twelve years ago, but it was not until 1932 that the final copy for the main table was prepared for the printer and approval for publication obtained. During this period manuscript tables, or rather tables printed by the adding machine that makes the final copy, have been in continual use in the Nautical Almanac Office; the table of tangents has proved invaluable in the formation of the right ascensions of Sun, Moon and planets."

The tables give the seven-figure natural values of the sine, cosine, tangent, and cotangent (p. 11–101) for every second of time in the quadrant, arranged semiquadrantally, with an auxiliary table (p. 7–10) of the function x^s cot x^s, to 3D or 8S, for every second to 30^m. The tables are intended for use with a calculating machine, and certain methods of direct and inverse interpolation can be used. The original 10D values were calculated with the aid of Andoyer's *Nouvelles Tables Trigonométrique Fondamentales* (*Valeurs Naturelles*) (v. 1–2, 1915–16), and Brandenburg's *Siebenstellige Trigonometrische Tafel* (second ed., 1931). In proof-reading checking was also done with no. (31).

In 1898 Henri de Sarrauton[19] proposed that the circumference be divided into 240 degrees, each hour being then divided into 10^d, and each degree into 10^m, each minute into 10 seconds, each second into thirds, etc. We are informed that M. Lebesgue, "eminent mathematician of Brussels," and his collaborator, M. Maurice Méry, actually calculated (35). *Tables des Fonctions Circulaires de la Division en* 240^d. I have not found that this ms. was ever published, nor can I identify M. Lebesgue.

Except for nos. (8), and (8C) in Library of Congress, and for certain items marked with "o," all printed tables mentioned in this article are (either in their original, filmed, or photostat form), in the library of Brown University. Originals of nos. (13), (25), and (29) are in the library of Harvard University, and of no. (33) in the U. S. Naval Observatory. I used a copy of no. (12) be-

longing to Mr. F. W. Hoffman of Pawtucket, R. I.; it does not appear to be in any of the larger libraries of America. R. C. A.

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[1] S. Günther, *Studien zur Geschichte der mathematischen und physikalischen Geographie. IV. Analyse einiger kosmographischer Codices der Münchener Hof- und Staatsbibliothek*, Halle, a/S., 1878, p. 249; codex p. 174^r–176^r: "Et notandum quod in praesenti tabula quilibet gradus et hora dividitur in 100 minuta, et quodlibet minutum in 100 secunda et sic de aliis." See M. Cantor, *Vorlesungen über Geschichte der Mathematik*, v. 2, 2nd ed., Leipzig, 1900, p. 185. Since the destruction of the library in March 1943, the codex may well be no longer in existence.

[2] *Trigonometria Britannica sive Doctrina Triangvlorvm Libri Dvo*, Gouda, 1633. vi p.; Trigon. Brit, lib. I, p. 1–60; Trig. Brit., lib. II, p. 61–110; table, 264 p. It was only the second book, on spherical trigonometry (50 p.) which was written by Gellibrand; all the rest was by Briggs; see Gellibrand's prefatory greetings to students of mathematics. Here, as usual, Delambre[9] is right. Unfortunately I first accepted as correct the statement of Glaisher (*Report*, p. 65) "The trigonometry is by Gellibrand." Hence the statements in *MTAC*, p. 10 (l. 18), 13 (l. 8), 26 (l. 1), should be modified accordingly, and the name Briggs be substituted for Gellibrand in the footnote on p. 13. Gellibrand tells us that Briggs' "Canon of Sines" was his first work, prepared "thirty years more or less" before, and that finally, tired out by the importunities of his friends to have it published, he decided to prepare it for the press, but had completed only the first book of the introduction at the time of his death. This statement suggests that Briggs may have computed the tables of natural sines, tangents and secants about 1600, and added the much less accurate logarithms of sines and tangents (see Delambre[9] and (10). Andoyer) perhaps about 1620, a few years after logarithms were first conceived. The interested reader will not overlook the tribute to Briggs in Delambre's "Rapport."[4]

[3] R. Mehmke, "Bericht über die Winkelteilung im Namen der Tafelcommission der Deutschen Mathematiker-Vereinigung," D. M. V., *Jahresb.*, v. 8, 1900, p. 139–158; see especially p. 145–146. Here are many valuable details supplementing the present article.

[4] Fuller information concerning matters here discussed may be found in the following sources: Riche de Prony, (a) "Notice sur les grandes tables logarithmiques et trigonométriques," and (b) "Éclaircissemens sur un point de l'histoire des tables trigonométriques," *Mémoires de l'Institut National des Sciences et Arts. Sciences Mathématiques et Physiques*, v. 5, Paris, 1803, p. 49–55, and 67–93.
J. B. J. Delambre, "Rapport sur les grandes tables trigonométriques décimales du Cadastre," *idem*, p. 56–66.
P. A. F. Lefort, (a) "Description des grandes tables logarithmiques et trigonométriques, calculées au Bureau du Cadastre sous la direction de Prony et exposition des méthodes et procédés mis en usage pour leurs construction," Paris, Observatoire, *Annales*, v. 4, 1858, Supplément, p. [123]–[150]; (b) "Note sur les deux exemplaires manuscrits des grandes tables logarithmiques et trigonométriques calculées au Bureau du Cadastre," Institut de France, Acad. d. Sc., *Comptes Rendus*, v. 46, 1858, p. 994–999. See also *Nouv. Ann. d. Math.*, v. 14, 1855, p. (14)–(17); and DeMorgan's article on Prony in *Penny Cyclopædia*, v. 19, 1841.

[5] J. W. L. Glaisher, "On logarithmic tables," R. Astr. So., *Mo. Notices*, v. 33, 1873, p. 455 footnote.

[6] See "List of logarithmic, trigonometrical, and astronomical calculations, in manuscript, by Edward Sang," p. 44–47 of E. M. Horsburgh, *Modern Instruments and Methods of Calculation*, London and Edinburgh, 1914. See also R. So. Edinb., *Proc.*, v. 9, 1878, E. Sang, "On the construction of the canon of sines, for the decimal division of the quadrant," p. 343–349; "On the precautions to be taken in recording and using the records of original computations," p. 349–352. V. 12, 1884, "On the construction of the canon of logarithmic sines," p. 601–619; v. 16, 1890, "Notice of fundamental tables in trigonometry and astronomy, arranged according to the decimal division of the quadrant," p. 249–256. See also *Proc.*, v. 28, 1908, p. 183–196.

[7] For grades, minutes and seconds we have used the notation g, `, ``. Fourteen other varieties of notations are exhibited by Mehmke[3] (p. 153). None of these are listed in Cajori's *History of Mathematical Notations* (1928–29).

[8] Institut de France, Acad. d. Sci., *Comptes Rendus*, v. 70, 1870, p. 1233–1236, 1390; and v. 71, 1870, p. 362–368. The astronomer Antoine Joseph Yvon Villarceau (1813–1883) to whom reference is here made, was the discoverer of the third series of circles on the torus ("Villarceau circles"; see "Théorème sur la tore," *Nouv. Ann. Math.*, v. 7, 1848, p. 345 ff. for an analytic proof, and F. G. M., *Exercises de Géométrie Descriptive*, fourth ed., Tours and Paris, 1909, p. 573 ff. for a geometric proof.)

[9] J. B. J. Delambre, *Histoire de l'Astronomie Moderne*, v. 2, Paris, 1821, p. 76–85, 393–420.

[10] Institut de France, Acad. d. Sci., *Comptes Rendus*, v. 126, 1898, p. 192–194. We are told that the decimal hour proposal was made in 1895 and that already decimal watches and decimal chronographs were available.

RECENT MATHEMATICAL TABLES

Details of recent tables are also to be found in our introductory article of this issue, nos. (33). Dieckvoss and Kox, (34). Comrie; and in **N4** Gifford and C. G. S. Tables, **N5** C. G. S. and Peters, **N6** Peters and Comrie, **N7** Smithsonian Tables.

89[D, E].—PROJECT FOR COMPUTATION OF MATHEMATICAL TABLES (A. N. Lowan, technical director). *Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments.* Prepared by the Federal Works Agency, Work Projects Administration for the State of New York, conducted under the sponsorship of the National Bureau of Standards. New York, 1940, 1939. xx, 405 p. 20.9×27 cm. Reproduced by a photo offset process. Sold by the U. S. Bureau of Standards, Washington, D. C. $2.00; foreign $5.00.

The main part of this volume (p. 1–400) is occupied with a table of the circular and hyperbolic sines and cosines for the range [0.0000 (0.0001) 1.9999; 9D]. Then follow supplementary tables: II (p. 402–403), circular and hyperbolic sines and cosines for the range [0.0(0.1)10.0; 9D]; and III (p. 404–405) a conversion table for radians and degrees, to 6D.

The expansions of the functions involved terms of the form $U_n(x)=x^n/n!$ The values of this function for the key arguments $x_0=0.01, 0.02, \cdots$ were computed to 15D for $n=0, 1, 2, \cdots p$, where p is such that $U_p(x_0)$ is not greater than one unit in the fifteenth place. The values of $U_n(x)$ for all other arguments were computed with the aid of a recurrence formula, successive applications of which involved self-checking. But the other checks applied included comparison of the values of sin x and cos x at intervals of 0.001 with the values given in Van Orstrand's Table, and in the New York Project's *Tables of Sines and Cosines for Radian Arguments*, computed by an independent method; see RMT **81**. The values of the hyperbolic functions were also checked by combining the values of e^x and e^{-x} in the New York Project's volume *Tables of the Exponential Function e^x*, for $x=[0.0000(0.0001)1,0000$; 18D], and [1.000(0.0001)25000; 15D], and $-[0.0000(0.0001)2.5000$; 18D]. After these tests, the resulting values of the circular and hyperbolic functions were rounded to 9D. The claim that an error in the ninth place does not exceed 0.51 is doubtless well founded.

From the fairly representative bibliography given in RMT **81**, it may be noted that so far as the circular functions are concerned, an appreciable portion of the present table covers new ground. Among other published tables of the hyperbolic sines and cosines, to 5D or more, and for real values of x, are the following (compare N **7**, IV):

THOMAS HOLMES BLAKESLEY (1847–1929), *A Table of Hyperbolic Cosines and Sines* . . . (Published by the Physical Society of London), London, 1890, 6 p. 15×24 cm. Cosh x and sinh x for $x=[0.01(0.01)4.00$; 7D]. Comparison of this table with only the values of the above mentioned Table II showed the following last figure errors in Blakesley: Cosh x—unit errors for $x=1.30$, 1.70, 2.10, 2.60, 2.90, 3.40, 3.60, 3.80, 3.90, and two units for $x=3.10$, and three units for $x=4.00$; sinh x—unit errors for $x=1.50$, 1.60, 1.80, 2.40, 2.60, 2.80, 3.10, 3.20, 3.30, 3.70, and two units for 3.50, 3.60, and 3.90.

JOHANN OTTO WILHELM LIGOWSKI (1821–1893), *Tafeln der Hyperbelfunktionen* . . . , Berlin, 1890, 16.4×24.6 cm. P. 58–61, 67–79, sinh x and cosh x, for $x=[0.00(0.01)2.00$; 6D], [2.00(0.01) 8.00; 5D], with differences.

ANGIOLO FORTI, *Nuove Tavole delle Funzioni Iperboliche* . . . , Rome, 1892. 16.4×23.9 cm. Sinh x and cosh x for $x=[0.0000(0.0001)2.000(0.0010)2.010$; 6D], [2.00(0.01)8.00; 5–7S]. Becker and Van Orstrand note that in these tables there are frequent errors of 1, 2, and 3 units in the last decimal place.

G. F. BECKER and C. E. VAN ORSTRAND, *Smithsonian Mathematical Tables. Hyperbolic Functions*, Washington, Smithsonian Institution, 1909; fifth reprint 1942, p. 88–171. 15×22.9 cm. Sinh x and cosh x for $x=[0.0000(0.0001)0.1000$; 5D], [0.100(0.001)3.000; 5D], [3.00(0.01)6.00; 4D].

K. Hayashi, *Fünfstellige Tafeln der Kreis- und Hyperbelfunktionen . . . mit natürlichen Zahlen als Argument.* Berlin and Leipzig, Gruyter, 1921. 16×23.1 cm. sin x, cos x, sinh x, cosh x, for x=[0(0.0001)0.100(0.001)3.00(0.01)6.3(0.1)10.0; 5D].

Ulfilas Meyer and Adalbert Deckert, *Tafeln der Hyperbelfunktionen. Formeln* Berlin, 1924, p. 6–17. 17×24.3. Sinh x and cosh x for x=[0.000(0.001)3.009; 5D].

J. R. Airey, Br. Ass. Adv. Sci., *Report*, 1926, p. 295–296. 21.5×27.9. Sinh x and cosh x for x=[0.1(0.1)10.0; 15D]; also in Br. Ass. Adv. Sci., *Mathematical Tables*, v. 1, London, B.A.A.S., 1931, Table VI, p. 30. 21.5×27.9 cm.

K. Hayashi, *Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen . . .*, Berlin, Springer, 1926, p. 13–201. 21×27.3. Sinh x and cosh x for x=[0.00000(0.00001)0:00100; 20D], [0.0010(0001)0.0999; 10D], [0.100(0.001)2.999; 10D], (3.00(0.01) 9.99; 10D], [10.0(0.1)20; 15D], [21(1)39; 15D], [39(1)50; 31–33S], p. 8–201. Also sinh(πx/360), and cosh(πx/360), for x=[0(1)360; 10D], p. 96–166 (alternate pages). Unreliable table. Since

C. E. Van Orstrand gave (Nat. Acad. Sci., Washington, *Memoirs*, v. 14, 1921, p. 40–45) a table for $e^{\pm(n\pi/360)}$ for n=[0(1)360; 23D], values for sinh (πx/360) and of cosh (πx/360), considerably more extensive than those of Hayashi, are readily found.

J. R. Airey, Br. Ass. Adv. Sci., *Report*, 1928, p. 308–316. 21.5×27.9. Sinh πx and cosh πx, for x=[0.00(0.01)4.00; 15D]; also in Br. Ass. Adv. Sci., *Mathematical Tables*, v. 1, London, B.A.A.S., 1931, Table V, p. 28–29. These tables "are required in the computation of the elliptic theta functions with imaginary argument, and of the gamma function with complex argument."

K. Hayashi, *Fünfstellige Funktionentafeln Kreis-, zyklometrische, Exponential-, Hyperbel-, . . . Funktionen . . .*, Berlin, Springer, 1930, p. 3–41, 60–64. 16.5×24.7 cm. sinh x and cosh x for x=[0.00(0.01)10.00; 5D]. Also sinh πx and cosh πx for x=[0.00(0.01)0.1(0.1)10.0; 5D], 7/6, 13/6, 19/6, 5/4, · · · , 7/2, · · · , 23/6 [21 miscellaneous values]. Unreliable table.

L. J. Comrie, table of sinh πx and cosh πx for x=[0.0000(0.0001)0.0100; 15D], Br. Ass. Adv. Sci., *Mathematical Tables*, v. 1, London, B.A.A.S., 1931, Table IV, p. 24–25; intended for use as an auxiliary table to Table V of Airey (1928).

F. E. Fowle, ed., *Smithsonian Physical Tables*, eighth rev. ed., first reprint, (*Smithsonian Misc. Coll.*, v. 88) Washington, Smithsonian Institution, 1934, p. 41–47. 14.9×22.8 cm. Sinh x and cosh x for x=[0.00(0.01)3.0; 5D], [3.0(0.1)5.0; 4D].

K. Hayashi, *Tafeln für die Differenzenrechnung sowie für die Hyperbel-, Besselschen, elliptischen und anderen Funktionen*, Berlin, Springer, 1933, p. 38–47. 21.2×27.7 cm. Sinh πx and cosh πx for x=[0.01(0.01)0.99; 6D], [1.00(0.01)10.00; 5–8S].

Samata Sakamoto, *Tables of Gudermannian Angles and Hyperbolic Functions*, Tokyo, 1934. 12.6×18.7 cm. Table III, p. 112–137 gives sinh x and cosh x for x=[0.000(0.005)0.100(0.010) 0.20(0.01)3.00(0.05)4.0(0.1)10.0; 5D]. Table IV, p. 138–199, has sinhπx, coshπx for x=[0.00(0.01)0.10(0.10)5.00(0.50)9.00, 10.00; 10D to 2.4, then 13S to 20S]. A comparison of this latter table for the range 0 to 4 in Airey's table of 1928 showed the following errors made by Sakomoto: for x=0.02 (cosh πx), and 0.40 (sinh πx) unit errors in the last place; x=0.80 (cosh πx) *for* 6.21314 32607 *read* 6.21314 32657; x=3.30 (sinh πx and cosh πx) *for* 13900.543 · · · *read* 15900.543 · · · .

J. R. Airey, assisted by L. J. Comrie, "The circular and hyperbolic functions, argument $x/\sqrt{2}$." *Phil. Mag.*, s. 7, v. 20, 1935, p. 721–726 and 726–731. sin ($x/\sqrt{2}$), cos ($x/\sqrt{2}$) and sinh ($x/\sqrt{2}$), cosh ($x/\sqrt{2}$), each for x=[0.0(0.1)20.0; 12D]. The calculations of these functions for x=2 to 20 were based on their values when x=1, which are given to 20D.

Further, C. A. Bretschneider, gave sinh 1 and cosh 1 each to 105D, *Archiv d. Math. U. Physik*, v. 3, 1843, p. 28–29.

We note also two other tables which may be regarded as sort of supplementary to the present list, as well as to the bibliography in RMT **81**. The first table is in

Vladimir Vassal, *Nouvelles Tables donnant avec Cinq Décimales les Logarithmes Vulgaires et Naturels des Nombres . . . et des Fonctions Circulaires et Hyperboliques pour tous les Degrés du*

Quart de Cercle de Minute en Minute, Paris, 1872, p. 67–111. For every sexagesimal minute there is a column giving the corresponding number of radians x to 5D, and another column for the values of u, to 5D, such that $\sinh u = \tan x$, $\cosh u = \sec x$, $\operatorname{sech} u = \cos x$, $\tanh u = \sin x$, etc. The corresponding values of the circular and hyperbolic functions to 5D, may then be read off. There are similar tables, for every sexagesimal, and centesimal, minute, in

L. Potin, *Formules et Tables Numériques relatives aux Fonctions Circulaires, Hyperboliques, Elliptiques*, Paris, 1925, p. 450–494, and 496–595.

Hence, not only in the Circular but also in the Hyperbolic Sines and Cosines of the volume under review there are important additions to previous ranges of values. The U. S. Bureau of Standards has performed very notable service, not only in making this, and a dozen other admirable volumes emanating from the New York group, available to scientists, but also at nominal charges.

R. C. A.

90[D].—Project for Computation of Mathematical Tables (A. N. Lowan, technical director), *Table of arctan x*. Prepared by the Federal Works Agency, Work Projects Administration for the State of New York, conducted under the sponsorship of the National Bureau of Standards. New York, 1942, xxv, 169 p. 20.9×27.1 cm. Reproduced by a photo offset process. Sold by the National Bureau of Standards, Washington, D. C. \$2.00; foreign \$2.50.

This volume gives values to 12 decimals of the definite integral

$$\arctan x = \int_0^x \frac{du}{1+u^2}$$

with second differences for the following ranges of x: 0(0.001)7, p. 2–71; 7(0.01)50, p. 72–114; 50(0.1)300, p. 115–139; 300(1)2000, p. 140–156; 2000(10)10000, p. 157–164.

As with all major tables published by this Project, this table is more extensive and accurate than any previously published table of its sort. The foreword to this volume, by W. G. Bickley, (at whose suggestion the table was produced) gives many instances of the utility of this table. The intimate connection of the function with the natural logarithm is stressed. For instance, armed with tables of these two functions, the computer may evaluate the integral of any rational function. Among the many applications of arctan x might have been mentioned the gudermannian

$$gdx = \arctan(\sinh x).$$

As pointed out in the foreword, this table is not intended to be used for ordinary trigonometry.

The introduction contains the usual formulas for arctan x including the well known infinite series of Gregory and Euler, and a discussion of the problem of interpolation. Besides the usual Everett's formula, using second differences, the following formula is available for this function:

$$\arctan x = \arctan x_0 + \theta - \theta^3/3 + \cdots$$

where $x = x_0 + h$, x_0 being a tabulated argument, and $\theta = h/(1 + xx_0)$. Inverse interpolation, by two methods, is not difficult, so that the table can be used to find tan y for y in radians.

At the end of the volume are auxiliary tables of $p(1-p)$ and $p(1-p^2)/6$ for use in interpolation together with two tables for converting degrees, minutes, and seconds to radians and vice versa.

A peculiar feature of the main table is the fact that the last 50 percent of it, in spite of the coarseness of the argument x near the end, is devoted to less than four percent of the range of the function arctan x. An alternative arrangement in which the range of x is 0(.0001)1 would have been sufficient in view of the relation

$$\arctan x = \frac{\pi}{2} - \arctan(1/x).$$

According to the introduction, the application of this formula for $x > 1$ "would generally be quite laborious, as it would involve finding the reciprocal of x, and then interpolating for that argu-

ment." The reviewer believes that with the argument in interval as fine as .001, the consequent saving in the labor of interpolation would more than offset the trouble of finding $1/x$. The present arrangement requires a second difference of as much as 649519 units in the twelfth decimal place near $x=1/\sqrt{3}$, so that linear interpolation is correct to only 6 decimal places. The alternative arrangement would have given 8 decimal places and, of course, simpler second difference interpolation, not to mention 100 instead of 164 pages. Problems for which the present table would seem to have been arranged, namely those involving the arctangent of large integers are not mentioned in the introduction or foreword nor are they known to the reviewer.

There is given (p. xxiii–xxv) a bibliography of 15 tables (mostly small) of arctan x and related functions together with a few errata.

There is every reason to believe that this excellent table, produced by subtabulation and checked to sixth differences, is as free from errors as are the other dozen major tables of this Project.

D. H. L.

91[K].—(i) WILLIAM FLEETWOOD SHEPPARD (1863–1936), *The Probability Integral* . . . Completed and edited by the Br. Ass. Adv. Sci., Committee for the Calculation of Mathematical Tables. Cambridge Univ. Press, publ. for the B.A.A.S., 1939. xi, 34 p. 21×28 cm. Portrait frontispiece of Sheppard. 8/6.

(ii) PROJECT FOR THE COMPUTATION OF MATHEMATICAL TABLES (A. N. Lowan, technical director), *Tables of Probability Functions*. Prepared by the Federal Works Agency, Work Projects Administration for the State of New York, conducted under the sponsorship of the National Bureau of Standards, New York. v. 1, 1941, xxviii, 302 p.; v. 2, 1942, xxi, 344 p. 20.9×27 cm. Reproduced by a photo offset process. Sold by the U. S. Bureau of Standards, Washington, D. C. \$2.00 + \$2.00; foreign \$5.00.

These works which are intimately related to one another, provide new and authoritative tables of what have been called the probability or error functions, that is to say the function Ae^{-kx^2}, where k may be either 1 or $\frac{1}{2}$, and its integral. The multiplier is usually chosen so that the integral over the infinite range is unity.

Although these functions have been the subject of interest for more than a century, there is as yet no standard notation for either of them. Perhaps one reason for this difficulty is found in the fact that while the statistician, a rather recent addition to the scientific fraternity, is interested in the function $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$ and its integral, the physicist finds more use for the function e^{-x^2} and its integral. Hence it would seem that each person who has encountered the functions anew, has given new symbols for each of their several forms. This same lack of uniformity is encountered in the volumes under review.

Sheppard in his work adopts the following notations, which have become fairly common in English works because of their use in *Biometrika*:

$$z_x = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2},\ \tfrac{1}{2}(1-\alpha_x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{1}{2}t^2}dt, \text{ that is to say,}$$

$$\alpha_x = \frac{1}{\sqrt{2\pi}}\int_{-x}^{x} e^{-\frac{1}{2}t^2}dt = \frac{2}{\sqrt{2\pi}}\int_0^x e^{-\frac{1}{2}t^2}dt, \qquad F(x) = \tfrac{1}{2}(1-\alpha_x)/z_x,$$

$$L(x) = -\log_e\left\{\tfrac{1}{2}(1-\alpha_x)\right\}, \qquad l(x) = \log_{10}\left\{\tfrac{1}{2}(1-\alpha_x)\right\}.$$

Before describing the contents of Sheppard's tables it will be necessary to explain the interpolation scheme which he employs. In most tables a difference method is used based upon the Gregory-Newton formula, or one of its variants, that is to say, the formula:

$$f(x+\theta h) = f(x) + \theta\Delta f(x) + \frac{\theta(\theta-1)}{2!}\Delta^2 f(x) + \cdots .$$

where h is the interval of the argument. One of the great advantages of this formula, or its variants such as the common Everett's formula which employs only central differences, is found in the fact that the value of h does not appear explicitly in it. Hence a computer can provide an interpolation scheme merely by giving a set of differences. The disadvantage is found in the binomial-coefficient multipliers of the successive differences, which must either be computed or read from tables.

Sheppard for his interpolation scheme decided to make use of the Taylor's expansion of a function, that is to say,

$$(x + \theta h) = f(x) + \theta f_1(x) + \theta^2 f_2(x) + \cdots + \theta^n f_n(x) + \cdots$$

where we employ the abbreviation $f_n(x) = \dfrac{h_n}{n!} f^{(n)}(x)$.

There is obviously some advantage gained in the ease of interpolation by replacing the binomial coefficients by powers, but it is also clear that the labor of computing the original table is very greatly increased since h appears explicitly in the multipliers and derivatives must be computed instead of differences. But when, as in the present tables, as many as 16 differences would be required were a full interpolation computed, it is readily seen that there is great advantage gained in the use of these so-called *reduced derivatives*.

Table I in Sheppard's work gives the values of $F(x)$ over the range $x=[0.00(0.01)10.00; 12D]$ together with the reduced derivatives $h^nF^{(n)}/n!$, n ranging from 5 in the early part of the table to 3 in the latter part.

Table II provides the values of $F(x)$ for $x=[0.0(0.1)10.0; 24D]$ together with the functions $h^nF^{(n)}/n!$, where n ranges from 16 in the early part of the table to 13 in the latter part.

Table III gives eleven values of $L(x)$ for $x=[0(1)10; 24D]$.

Table IV, gives the values of $L(x)$ over the range $x=[0.0(0.1)10.0; 16D]$, with corresponding values of $h^nL^{(n)}/n!$, n ranging from 10 to 8.

Table V provides values of $l(x)$, for $x=[0.0(0.1)10.0; 12D]$ together with the corresponding values of $h^nl^{(n)}/n!$ with n ranging from 7 to 6.

Table VI gives $l(x)$ for $x=[0.00(0.01)10.00; 8D]$, together with the central differences δ^2.

The following quotation from the "Introduction" indicates the method of computation and the accuracy of the tables.

"Table II was evidently constructed from Laplace's continued fraction and the derivatives calculated from its convergents. Table I was obtained by subtabulating Table II to interval (0.01). Sheppard used the fact that any tabular entry is the sum of the next tabular entry and its reduced derivatives, all taken positively; while the reduced derivatives of any entry are simple linear functions, with known coefficients, of these quantities, already calculated in Table II at interval (0.1). He never completed his subtabulation. This has been done on the Association's National machine by Mr. F. H. Cleaver using a method devised by Mr. D. H. Sadler. Mr. W. L. Stevens gave great help in supervising the calculations. All the tables have now been checked, Table III by recalculation, Tables II, IV and V by summing the function and its reduced derivatives for each value of the function; the latter process does not of course ensure the accuracy of the last figure. It is an example of the remarkable accuracy of Sheppard's work that not a single error was discovered in any of the entries in Table II."

The name of Sheppard is familiar to every student of statistics. He was an authority on methods of graduating data and most textbooks give an account of what is called the method of *Sheppard's corrections* for the adjustment of the values of moments computed from discrete data.

In the American volumes the following notations were adopted:

$$H'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}, \quad H(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt; \quad Q(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad P(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^x e^{-\frac{1}{2}t^2} dt.$$

The first volume is devoted to the evaluation of $H'(x)$ and $H(x)$, the second to $Q(x)$ and $P(x)$. Both volumes contain introductory explanations about the method of computation of the functions and the use of the tables.

Table I, v. 1, gives the values of $H'(x)$ and $H(x)$ for $x=[0.000(0.0001)1.0000(0.001)5.600; 15D]$. Since $H'(x)$ at $x=5.600$ is 27×10^{-15} and $H(x)$ is $1-2\times10^{-15}$, one page of values is given at the end of the table showing the change in the argument for each unit change in the fifteenth place of these tabulated values. Neither argument reaches 6.000.

Table II, v. 1, gives the values of $H'(x)$ and $1-H(x)$ for $x=[4.00(0.01)10.00; 8S]$. It is interesting to note that $H'(10)=4.1976562\times10^{-44}$ and $1-H(10)=2.0884876\times10^{-45}$.

The last page of v. 1 gives the constants π, $1/\pi$, $\sqrt{\pi}$, $1/\sqrt{\pi}$, $\sqrt{2\pi}$, $1/\sqrt{2\pi}$, $2/\sqrt{\pi}$, $\log \pi$, $\log \sqrt{2\pi}$, e, $\log e$, and $\log 2$, to 16D.

Table I, v. 2, gives the values of $Q(x)$ and $P(x)$ for $x=[0.0000(0.0001)1.0000(0.001)7.800; 15D]$. Since $Q(x)$ at $x=7.800$ is 25×10^{-15} and $1-P(x)$ is 6×10^{-15}, one page of values is given showing the argument which corresponds to one unit change in the fifteenth place in these tabulated values. $Q(x)$ is zero to 15D for $x=8.285$ and $P(x)$ is unity to 15D for $x=8.112$.

Table II, v. 2, provides values of $Q(x)$ and $1-P(x)$ for $x=[6.00(0.01)10.00; 7S]$. One may observe that the first significant figure for both $Q(x)$ and $P(x)$, when $x=10.00$, is in the 23rd place. These values are far beyond conceivable use in statistics since the realistic range for data seldom exceeds three standard deviations, and these tables extend the range to 10 standard deviations. However, one can never tell to what other uses such fundamental values may be adapted.

The method of computing the tables in both volumes was essentially the same and made use of the well known properties of the Hermite polynomials. As with the probability functions themselves, notations differ for the Hermite polynomials. It will be observed that the nth derivative of e^{-kx^2} is the product of e^{-kx^2} by a polynomial of nth degree. These polynomial multipliers are called Hermite polynomials after the French mathematician, Charles Hermite (1822–1901) who first studied them. Two forms are to be observed, those corresponding to $k=\frac{1}{2}$ and those corresponding to $k=1$. The writer prefers the notation:

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} e^{-\frac{1}{2}x^2}, \qquad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx} e^{-x^2}.$$

One may observe that the two forms are connected by the relationship

$$h_n(x) = 2^{-\frac{1}{2}n} H_n\ (x/\sqrt{2}).$$

It may also be proved that they satisfy the following recurrence relationships:

$$h_{n+2}(x) - xh_{n+1}\ (x) + (n+1)h_n(x) = 0, \qquad H_{n+2}\ (x) - 2xH_{n+1}(x) + 2(n+1)H_n\ (x) = 0.$$

Returning to the computational problem, let us first write Taylor's series in the form

$$f(x \pm ph) = f(x) \pm \frac{ph}{1!} f'\ (x) + \frac{(ph)^2}{2!} f''(x) + \cdots,$$

where h is the tabular interval. Then if p is set equal to 1 and $f(x)=H(x)$, we obtain the expansion

$$H\ (x \pm h) = h(x) \pm hH'(x) + \frac{h^2}{2!} H''\ (x) + \cdots \pm \frac{h^n}{n!} H^{(n)}(x) \pm \cdots.$$

To begin with $H'(x)$ was computed for the 60 "key arguments" $x=[0.0(0.1)6.0; 25D]$. By means of the properties of the Hermite polynomials, derivatives of higher order were next computed for the same arguments so that 25-decimal accuracy would be obtained in $10^{-k}H^{(k)}(x)/k!$

Beginning then with the expansion for $H(x\pm h)$ and noting that $H(0)=0$, it was easy to evaluate $H(0.1)$. From this new value $H(0.2)$ and $H(0.0)$ were computed, and then in succession $H(0.3)$ and $H(0.1)$, $H(0.4)$ and $H(0.3)$, etc. The second computation of each previously computed value was used as a check on the computations of the derivatives employed. This process was continued until $H(6.0)$ was obtained, which was then checked with its direct evaluation from the asymptotic expansion of $H(x)$. With these key values of $H(x)$ and $H'(x)$ subtabulation was then employed to complete the table, Taylor's series being used in the computation.

Because of effective tests applied to check the entries, including differencing and double proofreading, the authors believe that "this table is entirely free from error."

It seems unfortunate that neither of the major tables contain differences, particularly since fourth differences are negligible and δ^2 would have been sufficient. However, when derivatives of

the second and higher orders are neglected, the following easily applied formulas are available for interpolation:

$$H(x_0 + ph) = H(x_0) + phH'(x_0), \qquad H'(x_0 + ph) = H'(x_0)[1 - 2x_0ph].$$

Six-place accuracy or better is obtained by these formulas. Nevertheless, the publication of differences would have added greatly to the value of the tables and the omission is very much to be regretted in a work of such importance.

H. T. D.

92[K, L].—A. N. LOWAN, N. DAVIDS, A. LEVINSON, "Table of the zeros of the Legendre Polynomials of order 1–16 and the weight coefficients for Gauss' mechanical quadrature formula," Amer. Math. So., *Bull.*, v. 48, 1942, p. 739–743. 15.4×24.2 cm. The tables were computed by the Project for Computation of Mathematical Tables, conducted by the W.P.A., New York.

Legendre's polynomial of the nth order, or zonal surface harmonic of the first kind, may be defined by

$$z = P_n(x) = \frac{1}{2^n n!}\,\frac{d^n(x^2-1)^n}{dx^n},$$

which is a particular solution of Legendre's equation

$$(1 - x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + n(n+1)z = 0.$$

Zonal harmonics P_n were first introduced in 1784 by Adrian Marie Legendre (1752–1833), in a paper published in *Mémoires des Savants Étrangers*, v. 10, 1785, and applied to the determination of the attractions of solids of revolution. $P_n(x)=0$ has n distinct roots between -1 and $+1$, arranged symmetrically about $x=0$. While for small values of n the expression for P_n is comparatively simple, when n is as large as 16 we get

$$P_n(x) = (1/2^{15})(300\ 540\ 195x^{16} - 1\ 163\ 381\ 400x^{14} + 1\ 825\ 305\ 300x^{12} - 1\ 487\ 285\ 800x^{10} + 669\ 278\ 610x^8 - 162\ 954\ 792x^6 + 19\ 399\ 380x^4 - 875\ 160x^2 + 6435).$$

Finding the zeros of even just this P_{16}, to 15D, is no small piece of computation. The Project calculated such zeros by successive approximations, combining synthetic division with the Newton-Raphson method. They were checked by using the relations between roots and coefficients.

Gauss's method of mechanical quadrature was set forth in his paper published in *Commentationes Societatis Regiae Scientiarum Gottingensis Recentiores*, v. 3, 1816; also, with certain misprints not in the original article, in *Werke*, v. 3, 1866, p. 165–196. If $x_1, x_2, \cdots, x_n$ are the zero values of $P_n(x)$, Gauss stated, in effect, the theorem

$$\int_{-1}^{1} f(x)dx = a_1 f(x_1) + a_2 f(x_2) + \cdots + a_n f(x_n)$$

where $a_i = \displaystyle\int_{-1}^{+1} \frac{P_n(x)dx}{P_n'(x_i)(x - x_i)}$, and $f(x)$ is a polynomial of degree not greater than $2n-1$ (C. G. J. Jacobi, 1826). If $f(x)$ is not a polynomial, but is with its derivatives continuous within the range, we have an approximation to the quadrature which becomes closer as n increases.

The table before us contains the non-negative zeros and corresponding a's, $n=2, \cdots, 16$, to 15D. Let us see to what extent the results here are new. In that fine work of B. P. Moors, *Valeur Approximative d'une Intégralé Définie*, Paris, 1905, viii, 195 p. +11 folding plates, the non-negative zeros and a's are given for $n=2(1)10$; 16D. These values are to one more decimal place than the one under review; otherwise the tables are in agreement, except that for $n=9$ the fifteenth digits of a_0 and a_1 differ by 3–4 units each. Moors is here in error. B. deF. Bayly in

Biometrika, v. 30, 1938, p. 193, gave the zeros and a's for $P_{12}(x)$ to 13D. With his correct results we found two errors in the zeros of the table under review, as well as last figure errors of Bayly in six of the a's; Mr. Lowan pointed out another error in $P_{11}(x)$. See MTE 6. Thus there were no dependable results beyond the range of Moors, and Bayly, when the new table added the range $n=11, 13(1)16$; 15D.

Gauss's results for the interval (0, 1) are reproduced in somewhat modified form for the interval $(-1, +1)$, $n=[2(1)7$; 16D], in Heine, *Kugelfunktionen*, v. 2, Berlin, 1881, p. 15–16 (the last three figures here in each of the zeros for $n=4$ are erroneous); these were copied from Heine, with the errors, in E. W. Hobson, *Spherical and Ellipsoidal Harmonics*, Cambridge, Univ. Press, 1931, p. 80–81. In H. J. Tallquist, *Grunderna af Teorin för Sferiska Funktioner, jamte Användningar inom Fysiken*, Helsingfors, 1905, p. 400 are given the zeros for $n=2(1)8$; 7D (in $n=8$ *for* 0.9602898, *read* 0.9602899). For the interval $-1/2$ to $+1/2$ E. J. Nyström calculated the zeros and a's $n=2(1)10$; 7D, *Acta Math.*, v. 54, 1930, p. 191. Nyström notes (p. 190) an error in Gauss, for $n=2$, where in place of $a''=1{,}887\cdots$, should be $a''=0{,}887\cdots$.

For a discussion and comparison of different methods of mechanical quadrature see especially the work of Moors, with an excellent historical survey, referred to above; but *Tract for Computers*, no. X, contains a very brief treatment by J. O. Irwin, *On Quadrature and Cubature or On Methods of Determining Approximately Single and Double Integrals*, Cambridge, 1923.

R. C. A.

I have already had occasion to use this table to good effect. Readers who are unacquainted with the literature of the subject and who may wish to use the table will, however, be confused by the fact that the description of the table does not agree in its notation with that of the table itself, and may be misled by the fact that the formula given for the remainder is seriously incorrect. This note is an attempt to make this valuable table just a little more useful.

The subscript i used above runs, in the table, from 1 to n when n is even, and from 0 to $n-1$ when n is odd. Since the roots x_i are symmetrically situated with respect to the origin, the table gives only the non-negative roots

$$x_1, x_2, x_3, \cdots, x_{n/2} \quad \text{if } n \text{ is even,}$$
$$x_0, x_1, x_2, \cdots, x_{(n-1)/2} \quad \text{if } n \text{ is odd,}$$

arranged in order of increasing magnitude. Opposite each such x is found the corresponding a. The other x's and a's are given by

$$x_{n-i} = -x_i, \quad a_{n-i} = a_i, \qquad i = 1, 2, \cdots, n-1, \text{—if } n \text{ is odd;}$$
$$x_{n-i+1} = -x_i, \quad a_{n-i+1} = a_i, \qquad i = 1\ 2, 3, \cdots, n, \text{—if } n \text{ is even.}$$

With these notations Gauss' formula becomes

$$\int_p^q f(x)dx = \tfrac{1}{2}(q-p)\sum_{i=\epsilon}^{n+\epsilon-1} a_i F(x_i) + R_n(f)$$

where $\epsilon=0$ or 1 according as n is odd or even and where

$$F(u) = f\left(\frac{p+q}{2} + \frac{q-p}{2}\right)u.$$

As for the remainder term $R_n(f)$, the authors give

$$R_n(f) = \frac{f^{(2n)}(\xi)}{(2n+1)!}(q-p) \qquad (p \leq \xi \leq q) \tag{1}$$

whereas the correct formula, due to Markov and Mansion,[1] is

$$R_n(f) = \frac{(n!)^4 f^{(2n)}(\xi)}{[(2n)!]^2(2n+1)!}(q-p)^{2n+1}. \tag{2}$$

It is clear that when the range of integration is large (1) will be much too small. Since

$$\frac{2(n!)^4 4^{2n}}{\pi(2n)!(2n+1)!} = \frac{\int_0^{\pi/2} \sin^{2n+1}\theta d\theta}{\int_0^{\pi/2} \sin^{2n}\theta d\theta} < 1$$

a somewhat simpler form of (2) may be given as follows

$$(3) \qquad |R_n(f)| < \frac{2\pi|f^{(2n)}(\xi)|}{(2n)!}\left(\frac{q-p}{4}\right)^{2n+1} \sim \frac{\sqrt{8\pi n}}{e}\left\{\frac{(q-p)e}{8n}\right\}^{2n+1}.$$

Since we are concerned, after all, with very moderate values of n, it is perhaps more to the point to write (2) in the form

$$R_n(f) = \frac{(q-p)^{2n+1}f^{(2n)}(\xi)}{(2n)!U_n}$$

and to tabulate[2] the integers U_n as follows

n	U_n	n	U_n	n	U_n
2	180	7	176679360	12	182811491808400
3	2800	8	2815827300	13	2920656969720000
4	44100	9	44914183600	14	46670906271240000
5	698544	10	716830370256	15	745904795339462400
6	11099088	11	11445589052352	16	11922821963004219300

A reference should have been made to an important note by Uspensky[3] on the subject of $R_n(f)$. He shows that $R_n(f)$ possesses an asymptotic development in every way comparable with the celebrated Euler-Maclaurin formula which latter may be thought of as giving the asymptotic development of the remainder of the trapezoidal rule.

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[1] A. A. Markov, *O Nekotorykh Prilozheniiakh Algebraicheskikh Nepreryvnykh Drobeĭ.* [On some Applications of Algebraic Continued Fractions], Doctoral diss., St. Petersburg, 1884, p. 68; A. A. Markov, "Sur la méthode de Gauss pour le calcul approche des intégrales," *Math. Annalen*, v. 25, 1885, p. 429; and P. Mansion, "Détermination du reste dans la formule de quadrature de Gauss," Acad. Royale d. Sci. d. Lettres. et d. Beaux-Arts de Belgique, *Bulletins*, s. 3, v. 11, 1886, p. 303. Also in A. A. Markov, *Differenzenrechnung*, Leipzig, 1896, p. 68; Gauss's numerical results are given on p. 70.

[2] These values up to U_7 were given by Gauss, *Werke*, v. 3, p. 193–195.

[3] "On an expansion of the remainder in the Gaussian quadrature formula," Amer. Math. So. *Bull.*, v. 40, 1934, p. 871–876.

D. H. L.

93[I].—(i) A. N. LOWAN, H. E. SALZER, A. HILLMAN, "A table of coefficients for numerical differentiation," Amer. Math. So., *Bull.*, v. 48, Dec., 1942, p. 920–924.

(ii) W. G. BICKLEY, J. C. P. MILLER, "Numerical differentiation near the limits of a difference table," *Phil. Mag.*, s. 7, v. 33, Jan., 1942, p. 1–14 +4 folding plates.

Numerical differentiation presents two problems depending on whether (a) the given values of the function are known to a high degree of precision, as would be the case, for example, if one wishes to find the second derivative of the gamma function from a six place table of that function, or (b) the values are determined by experiment and are subject to considerable uncertainty.

Well known solutions of problem (a) which date back to Newton depend on interpolation to the given values by means of a polynomial of arbitrary degree n. The final result is an expression for $f^{(m)}(x)$ as a linear combination of successive differences of the function values:

$$\omega^m f^{(m)}(x) = \sum_{q=m}^{n} A_{m,q}\Delta^q f,$$

where ω is the tabular interval, $\Delta^q f$ is the q-th difference found from tabular values at the equally spaced points $x_1^{(q)}, x_2^{(q)}, \cdots, x_{q+1}^{(q)}$, and the coefficients $A_{m,q}$ depend only on the position of x relative to $x_1^{(q)}, x_2^{(q)}, \cdots, x_{q+1}^{(q)}$. Various particular methods are obtained by varying this relative position. In case these points are symmetrical about x, one gets a formula in terms of "central

differences." Central difference formulas are usually preferred whenever they can be used, but they are inapplicable to points at or near the ends of the table. In such a case one ordinarily uses the formula in terms of "forward differences," where $x_1^{(q)} = x$. Paper (i) is concerned with the calculation of the coefficients $A_{m,q}$ for this case, and gives a table of these coefficients for the first 20 derivatives up to those of the 20th difference; the coefficients of the first 12 of these had been already given in one of the tables of paper (ii).

Paper (ii) raises an objection to the use of advancing differences when x is not at the end of the table but is too near for central differences, on the grounds that forward differences do not use all the available tabular values nearest x. To meet this objection the authors have developed new general formulas in terms of "mixed differences," in which x can have any tabular position from $x_1^{(q)}$ to $x_{q+1}^{(q)}$. These formulas include central and backward and forward differences as special cases. In applications one uses central differences for derivatives of orders up to the first for which $x_1^{(q)}$ (or $x_{q+1}^{(q)}$) reaches the limit of the table; for higher derivatives the mixed differences for which $x_1^{(q)}$ (or $x_{q+1}^{(q)}$) is at the end of the table. Analogous formulas are also obtained for x midway between two tabular points; these enable one to subdivide the given tabular interval. The authors have computed the coefficients $A_{m,q}$ of these mixed difference formulas for differences up to those of the 12th order for the first 4 derivatives when x itself is a tabular point; also for the first three derivatives when x is midway between two tabular points. In all these cases the coefficients are rational numbers and their values are given exactly. The tables displaying these coefficients have an unusually convenient arrangement. The central differences from a horizontal line along the middle of the page with the mixed differences arranged on either side to form a triangular array. In a particular problem the values to be taken from the table will occur first along this horizontal line and then along the diagonal. Arrows are placed in the table to help guide the eye along the proper diagonal. Illustrative examples in the use of the tables are also included.

In problem (b), the methods given in these papers are unsuitable, in general. In fact no method based on the theory of interpolation, in which an approximating curve is passed through the given values, is suitable; for, to force the approximating curve to pass exactly through the given values is not desirable and will usually introduce wide oscillations in the derivatives. Some other method of approximation having the effect of smoothing out of the given data must be used in this case.

Among references to topics in paper (i) are the following: L. M. Milne-Thomson, *The Calculus of Finite Differences*, London, Macmillan, 1933, Chap. 7, p. 157–159; H. T. Davis, *Table of the Higher Mathematical Functions*, Bloomington, Ind., v. 1, 1933, p. 73–77; E. T. Whittaker and G. Robinson, *The Calculus of Observations, A Treatise on Numerical Mathematics*, 3d ed. London, Blackie, 1940, p. 62–65. The references in paper (ii) are to K. N. Bradfield and R. V. Southwell, "Relaxation methods applied to engineering problems. I—the deflexion of beams under transverse loading," R. So. London, *Proc.*, v. 161A, 1937, p. 155–181; L. J. Comrie, *Interpolation and Allied Tables*, London, H. M. Stationery Office, 1936. (Reprinted from the *Nautical Almanac for 1937*.), D. C. Fraser, "On the graphic delineation of interpolation formulae," Inst. Actuaries, *Jn.*, v. 43, 1909, p. 235–241; J. F. Steffensen, *Interpolation*, Baltimore, Williams & Wilkins, 1927.

P. W. KETCHUM

94[A, D, E].—H. S. UHLER, "A new table of reciprocals of factorials and some derived numbers," Connecticut Acad. Arts and Sci., *Trans.*, v. 32, 1937, p. 381–434. 16.2×24.4 cm. Compare RMT **86.**

The main table may be regarded as the superposition of two tables one of which is limited (for $n>14$) to 475D, so that it terminates with 1/214!, and the other is defined by 70S, and includes all values of $1/n!$ from $n=1$ to $n=369$. The upper limit was arbitrarily chosen so that the table would be adequate for the evaluation of e^{100} to about 100S. D. H. Lehmer gave the value of e to 707D (*Amer. Jn. Math.*, v. 48, 1926, p. 139–143), in order to match the 707-place value of π found by Shanks before 1874. H. S. Uhler reprints Lehmer's value and shows that it is in agreement with his own to 478D.

Other results given are e^{-1} (to 477D), e^2 (to 257D), e^4 (to 256D), e^6 (255D), e^8 (255D), e^{10} (253D), e^{-10} (258D), e^{100} (117D), e^{-100} (40S), sin 1 (477D), cos 1 (477D), sin 10 (212D), cos 10 (212D), cos 20 (212D), sin 100 (72D), cos 100 (72D), sin 200 (72D), cos 200 (72D).

The elaborate tests applied for checking the accuracy of the calculations are fully described, and tend to inspire unlimited confidence in the results.

The work of H. S. Uhler has checked the accuracy of the following earlier substantial results: C. A. Bretschneider, *Archiv d. Math. u. Phys.*, v. 3, 1843, p. 27–34, e, e^{-1}, sin 1, cos 1, sinh 1, cosh 1, each to 105D.

J. W. L. Glaisher, Cambridge Phil. So., *Trans.*, v. 13, 1883, p. 247. $1/n!$, $n=13(1)50$, to 28D. For $n=20$, 27, 41 and 50 "Glaisher's numbers end with 9, 7, 5, and 6 instead of 8.436, 6.974, 4.449, and 5.468 respectively."

C. E. Van Orstrand, Nat. Acad. Sci., *Memoirs*, v. 14, no. 5, 1921. $1/n!$, $n=1(1)74$, to 108D, p. 12–13; e, e^2, e^4, e^6, e^8, e^{10}, each to 42D, p. 16–17; e^{-10} to 52D, p. 27; e^{-100} to 19D, p. 28; sin 10, cos 10, cos 20, cos 100, sin 100 each to 23D, p. 47–48.

J. T. Peters and Johann Stein (1871–   ), *Anhang mathematischer Tafeln* in *Zehnstellige Logarithmentafel*, v. 1, Berlin, 1922. e^{-1} to 72D, p. 12; e^2, e^4, e^6, e^8, and e^{10} each to 32S, p. 12; sin 1, cos 1, each to 52D, p. 60.

R. C. A.

95[A, C].—H. S. Uhler, "Log π and other basic constants," Nat. Acad. Sci., *Proc.*, v. 24, 1938, p. 23–30. 17.5×25.8 cm.

The Table here includes the following results all most carefully checked: (1/2) log 2π and log π, each to 214D; ln π to 213D; log 2, log 3, log 5, log 7, log 17, each to 230D; ln 17 to 224D; ln 71 to 213D; log 71 to 110D; ln 113 to 213D; log 113 to 110D; π^{-1} to 253D; π^2 to 261D. Errors in Parkhurst's tables are noted. The value of π^{-1} was calculated by J. W. Wrench, Jr.; the most extensive earlier correct value was by G. Paucker, to 137D, *Archiv Math. Phys.*, v. 1, 1841, p. 1. π^2 was given correctly to 217D by S. Z. Serebrennikov, Akademiiâ Nauk, *Classe physico-mathématique, Mémoires*, s. 8, v. 19, 1906, p. 4.

James Stirling and Abraham De Moivre by different methods arrived at the following remarkable and very useful approximation[1] when x is large: $x! \approx (2\pi)^{\frac{1}{2}}(x)^{\frac{1}{2}}(e)^{-x}x^x$. De Moivre gave, in effect, the expansion

$$\ln x! = (1/2)\ln 2\pi + (x+1/2)\ln x - x + \frac{B_1}{1\cdot 2}\cdot\frac{1}{x} - \frac{B_2}{3\cdot 4}\cdot\frac{1}{x^3} + \frac{B_3}{5\cdot 6}\cdot\frac{1}{x^5} - \cdots + R$$

where $B_1, B_2, B_3, \cdots$ denote the Bernoulli numbers. Log $x!$ for $x=1(1)3000$, to 33D may be read off from F. J. Duarte, *Nouvelles Tables de log x!*, Geneva and Paris, 1927; logarithms to 61D for all numbers to 100 and of primes from 100 to 1100 may be found in Sharp's table (1717), and logarithms to the base e and to 48D, in Wolfram's table (1778); the Bernoulli numbers B_i up to $i=110$ are known; also log π to 61D (Sharp, 1717); and ln π to 48D in J. T. Peters and J. Stein, *Anhang mathematischer Tafeln* in *Zehnstellige Logarithmentafel*, v. 1, Berlin, 1922, p. 1. Hence computations in this connection to more than about 60D, for $x>1100$, call for further basic values. For this purpose H. S. Uhler has now provided (1/2) log 2π, log π, ln π, and log 2, while J. C. Adams gave ln 2, all to over 200D (1878 and 1887).

The formula underlying the calculation of ln π was based on an approximation for π due to Ramanujan (*Quart. Jl. Math.*, v. 45, 1914, p. 366; and *Collected Papers of Srinivasa Ramanujan*, Cambridge, 1927, p. 35), which, with a correction factor f making the formula exact, is as follows: $\pi=(355/113)(1-.0003/3533)f$, a rough value for $1-f$ being 347×10^{-18}. Hence the actual computation of ln π reduced to that of ln of 71,113, $1-.0003/3533$ and f. It was shown that the computation of ln 71 and ln 113 could be made to depend wholly on ln 2, ln 3, ln 5, ln 7, already calculated, and on certain rapidly converging series.

R. C. A.

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[1] J. Stirling, *Methodus Differentialis*, London, 1730, p. 137; second ed., 1764, p. 137; English edition by F. Holliday, 1749, p. 121. A. de Moivre, *Approximatio ad Summam Terminorum Binomii* $(a+b)^n$ *in Seriem expansi*, London, 1733; rev. transl. in A. de Moivre, *Doctrine of Chances*, London, second ed., 1738, p. 235–242; third ed., 1756, p. 243–250; for a facsimile of the 1733 publication see R. C. Archibald, "A rare pamphlet of Moivre and some of his discoveries," *Isis*, v. 8, 1926, p. 677–683. See also C. Tweedie, *James Stirling* . . . , Oxford, 1922, p. 119, 203–205.

96[A, C, D, E].—H. S. Uhler, "Recalculation of the modulus and of the logarithms of 2, 3, 5, 7 and 17," Nat. Acad. Sci., *Proc.*, v. 26, 1940, p. 205–212. 17.5×25.8 cm.

In the calculation of the table in RMT **95** the series

$$\ln\frac{p}{q} = 2\left\{\frac{p-q}{p+q} + \frac{1}{3}\left(\frac{p-q}{p+q}\right)^3 + \frac{1}{5}\left(\frac{p-q}{p+q}\right)^5 + \cdots\right\} = 2S[(p-q)/(p+q)],$$

with $p-q=1$, played an important role. With $p=5041=71^2$, $5040=2^4\cdot 3^2\cdot 5\cdot 7$

$$\ln 71 = 2\ln 2 + \ln 3 + (\ln 5 + \ln 7)/2 \qquad S(1/10081).$$

Similarly for $p=226$, ln 113 involves $S(1/451)$. Thus in the present paper, we have $S(1/5)$, $S(1/239)$, $S(1/2449)$, $S(1/4999)$, and $S(1/8749)$, in connection with ln 2, ln 3, ln 5, ln 7 and ln 17. J. C. P. Adams calculated the first four of these to 262D (1878 and 1887); see MTE **8**. These are here extended, with certainty on the author's part, to 328D. The values are also given of the following: arctan (1/451) to 215D; arctan (1/577) to 335D; arctan (1/2449), arctan (1/4999), and arctan (1/8749) each to 330D; and arctan (1/10081) to 216D.

Adams found M correct to 271D (1887). From his own ln 2 and ln 5 Uhler determined M, correct to 328D.

Five other values found in RMT **94** are here extended, viz: e^{10} to 289D; e^{-10} to 293D; and sin 10, cos 10, cos 20, each to 284D. These latter ranges are also supplementary to results in RMT **81**.

R. C. A.

97[A, K].—H. S. Uhler, "The coefficients of Stirling's series for log $\Gamma(x)$," Nat. Acad. Sci., *Proc.*, v. 28, 1942, p. 59–62. 17.5×25.8 cm.

When n is a positive integer, the asymptotic series of RMT **95** becomes

$$\ln\Gamma(x) = (1/2)\ln 2\pi + (x+1/2)\ln x - x + \sum_{m=1}^{\infty}(c_m/x^{2m-1}) + R,$$

where $c_m \equiv (-1)^{m-1}B_m/[(2m-1)(2m)]$. The table of the paper contains the first 71 values of c_m, many of which have recurring periods within the range of the table; c_{18} is given to 103S. Values of 100! to 158S, and of ln (100!) to 156S, are also given.

MATHEMATICAL TABLES—ERRATA

In this issue we have referred to Errata in RMT **89** (Blakesley, Forti, Hayashi, Sakamoto), RMT **92** (Lowan *et al.*, Moors, Bayly, Gauss, Heine, Hobson, Tallquist), RMT **94** (Glaisher), RMT **95** (Parkhurst, Serebrennikov), UMT **2** (Airey), N **4** (Gifford, C. G. Survey), N **5** (C. G. Survey), N **6** (Gifford), and in the first article of this issue (Callet, Brandicourt and Roussilhe, Jordan, Service Géog. 1914).

5. U. S. Coast and Geodetic Survey, Special Publication, no. 231, *Natural Sines and Cosines to Eight Decimal Places*, 1942; see RMT **77**.

End-figures are missing cos 1°44′41″ and 42″, namely: 0 and 5 respectively.

L. J. C.

Sin 36° *for* 0.587 78255, *read* 0.587 78525.

F. W. Hoffman, 689 East Ave., Pawtucket, R. I.

6. A. N. Lowan, N. Davids, A. Levenson, "Table of the zeros of the Legendre polynomials," 1942; see RMT **92**.

$n=11$, *for* $x_2=0.519096129110681$ *read* $x_2=0.519096129206812$
$n=12$, *for* $x_1=0.125333408511469$ *read* $x_1=0.125233408511469$
$n=12$, *for* $x_2=0.367831498918180$ *read* $x_2=0.367831498998180$

A. N. Lowan, and R. C. A.

7. J. T. Peters, *Zehnstellige Logarithmentafel, Erster Band, Zehnstellige Logarithmen der Zahlen von 1 bis 100 000 nebst einem Anhang mathematischer Tafeln*. Berlin, 1922. All of the errors noted below are in the "Anhang," arranged and calculated by J. T. PETERS and J. STEIN, p. i–xxviii, 1–195.

Table 3, p. 47, $1/42^8$, groups 5 and 6

for 85453 21863 *read* 85452 31863

It would seem that the check described in the Introduction (sum of the $1/42^n$, to 32D=1/41) must have been applied in ms., not in proof. The columns involved satisfy this check as amended but not as printed. P. VII Stirling's series for log $n!$, *for* $\frac{B_3}{5\cdot 6\cdot n}$ *read* $\frac{B_3}{5\cdot 6\cdot n^5}$

C. R. Cosens, Engineering Laboratory, Cambridge, England, Nov., 1941

Table 13, contains, mainly ln$N$, $N=2(1)146$, and all following prime numbers to 9973. This Table was computed by one of the great calculators of logarithms, J. Wolfram, Lieutenant of the Dutch artillery. After six years of intense application he computed ln N, $N=1(1)10009$, to 48D. They were first published in J. C. Schulze, *Recueil de Tables Logarithmiques, Trigonometriques et autres nécessaires dans les Mathématiques Pratiques*, [also t. p. in German], v. 1, Berlin, 1778, p. 189–259. Space is left for the logarithms of six numbers 9769, 9781, 9787, 9871, 9883, 9907 which Wolfram had, up to 1778, been prevented from computing by a serious illness. These were supplied two years later by Schulze in *Berliner Astronomisches Jahrbuch für das Jahr 1783*, Berlin, 1780, p. 191, as given to him by Wolfram, and also, from an independent calculation by Barzellini, "Oberbuchhalter der Grafschaften Görz und Gradisca." There have been various reprints or revisions of Wolfram's table; the first of these was in G. Vega, *Thesaurus Logarithmorum Completus*, Leipzig, 1794; this is the basis of the table of P. & S., in which there are at least the following ten errors.

	N	pentad	for	read	first discovered by
1.	829	4	67458	97458	Escott
2.	1087	10	598	597	Cosens
3.	1409	4	21666	21696	Gray
4.	3967	6	91589	91389	Duarte
5.	6343	3	23897	33897	Steinhauser
6.	7247	7	24102	25102	Duarte
7.	8837	4	42054	42354	Duarte
8.	8963	7	38152	38153	Duarte
9.	9623	4	83304	83305	Duarte
10.	9883	10	194	193	Cosens

No. 10 was given correctly by Wolfram but incorrectly by Vega and P. & S.; compare *Scripta Mathematica*, v. 3, 1936, p. 99–100, and v. 4, 1937, p. 293. There are two cases where Wolfram and Vega (1794 and 1923 reprint) were wrong, while P. & S. were correct, viz: ln 1087 (pentad 6), and ln 3571 (pentad 4). E. B. Escott communicated the result in no. 1 to L. J. Comrie in 1924.

Nos. 2 and 10 were found by C. R. Cosens in 1939, after recalculating the logarithms to 55D, assuming the accuracy of Grimpen's ln of primes to 127, to 82D.

No. 3 was indicated in Peter Gray, *Tables for the Formation of Logarithms & Anti-Logarithms to twenty-four or any less number of Places*, London, 1876, p. [39].

No. 5 was given in Anton Steinhauser, *Hilfstafeln zur präcisen Berechnung zwanzigstelliger Logarithmen zu gegebenen Zahlen und der Zahlen zu 20 stelligen Logarithmen*, Vienna, 1880, p. 1.

Nos. 4, 6, 7, 8, 9, were given by F. J. Duarte, *Nouvelles Tables Logarithmiques à 36 Décimales*, Paris, 1933, p. XXII.

R. C. A.

In Table 1, p. 7, M, the modulus of common logarithms is given to 282D. There are at least 14 incorrect digits among the last 19, viz.: 47 48049 05993 55353 05.

In Table 13, p. 152, ln 2, ln 3, ln 5, ln 7 are each given to 272 places of decimals and the last 9 or 10 digits in each are erroneous as follows:

in	for	read
ln 2	81 06850 15	70 95326 37
ln 3	92 45403 15	75 60690 11
ln 5	604 17624 80	580 59722 57
ln 7	210 03537 95	183 10810 25

All five of the values as printed by P. & S. were taken from a paper by J. C. Adams, Royal So. London, *Proc.*, v. 27, 1878, p. 92–93. Corrections were given by Adams in *Proc.*, v. 42, 1887, p. 24–25. See also J. C. Adams, *Scientific Papers*, Cambridge, v. 1, 1896, p. 464, 469–477. Adams states that M is now true certainly to 272 and probably to 273D.[1]

I have recently evaluated ln 127 to about 104 places to test the illustrative value given by P. & S. p. XXVII. The last figure [82nd] should be 7 instead of 4. This finding agrees very well with the comment on p. XXVIII which reads: " . . . ; die Endziffern des so bestimmten ln 127 weichen nur um 4 Einheiten von dem vorher erhaltenen Werte ab."

P. & S. apparently failed to compare their (?) 61-place Table 14b of ordinary logarithms (p. 156–162) with the appropriately abbreviated 84-place mantissas quoted from A. Grimpen (p. XXV). For the numbers 31, 43, 47, and 59 the difference (P. & S. minus Grimpen) equals $+0.8$, -1.0, -0.51, and $+1.35$ respectively. Nevertheless P. and S. state (p. III) that they tried to attain an accuracy "of half a unit in the last decimal place." In Table 14b, p. 158, ln 227 is incorrect; in the last 6 digits *for* 49465 6, *read* 49565 6.

H. S. UHLER, Dept. of Physics, Yale University
New Haven, Conn., Oct. and Nov. 1936

"A Table of the Common or Brigg's Logarithms for all Numbers to 100; and all Primes, to 1100, true to sixty one Figures" was first given in a work by Abraham Sharp (1651–1742), *Geometry Improved* . . . , London, 1717, p. [56]–[60]. It has been reprinted many times, for example, in the first stereotyped edition of François Callet, *Tables Portatives de Logarithmes*, Paris, 1795, and on to the 1899, and possibly later editions. P. & S. copied their table from Callet's work. Sharp gave also log π correct to 61D (p. 36–37).—EDITOR.

I now report the following 37 other errors of P. & S., who boast of last figure accuracy (p. III, l. 11–12):

page	no.	for last figures	read	abbreviation o
XXIII and 1	log π	6	5	4 999
XXIV	ln 23	81	82	81 984
XXIV	ln 41	59	60	59 623
XXIV	ln 59	73	74	73 593
XXIV	ln 61	11	12	11 854
XXV	ln 71	59	60	59 662
XXV	ln 73	32	33	32 763
XXV	ln 97	54	53	53 422
XXV	ln 103	97	96	95 917
XXV	ln 107	64	66	65 573
XXV	log 17	5795	5796	5795 684
XXV	log 71	7501	7500	7499 931
XXV	log 101	0771	0770	0770 238
XXV	log 113	6837	6838	6837 823

[1] H. S. Uhler's recent researches have shown that even this statement concerning M is not absolutely correct; the substitution in the *Anhang* for the 264th to the 282nd digits should be as follows:

for 47 48049 05993 55353 05 *read* 53 83562 22813 95603 05.

Adams's new 271st to 277th digits were 21868 25 so that his 272nd digit should be "2," not "1"; see RMT **96**. J. W. L. Glaisher in his article on "Logarithms," in the ninth edition of the *Encyclopædia Britannica* (1882), had the incorrect value of M given by Adams in 1878, without the statement of Adams at that time that he did not claim his value to be correct beyond 262D or 263D. The corrected value of 1887 (but still slightly incorrect, as we have seen) is in the eleventh edition of the *Britannica* (1911).—EDITOR

151	$\ln(1-9\cdot10^{-4})$	485	486	485 507
152	$\ln(1+8\cdot10^{-4})$	567	566	566 326
151	$\ln(1-7\cdot10^{-4})$	859	860	859 672
151	$\ln(1-5\cdot10^{-4})$	785	786	785 574
152	$\ln(1+5\cdot10^{-4})$	340	339	339 355
151	$\ln(1-2\cdot10^{-4})$	811	810	810 371
151	$\ln(1-1\cdot10^{-4})$	734	735	734 571
152	$\ln(1+1\cdot10^{-4})$	402	401	401 071
151	$\ln(1-8\cdot10^{-5})$	613	614	613 570
152	$\ln(1+8\cdot10^{-5})$	796	797	796 578
151	$\ln(1-6\cdot10^{-5})$	899	898	898 045
151	$\ln(1-5\cdot10^{-5})$	846	845	845 433
152	$\ln(1+5\cdot10^{-5})$	980	981	980 850
151	$\ln(1-4\cdot10^{-5})$	446	445	445 439
151	$\ln(1-3\cdot10^{-5})$	774	773	773 336
151	$\ln(1-1\cdot10^{-5})$	682	683	682 540
151	$\ln(1-9\cdot10^{-6})$	599	597	597 457
151	$\ln(1-8\cdot10^{-6})$	358	357	357 389
151	$\ln(1-7\cdot10^{-6})$	606	605	604 608
151	$\ln(1-5\cdot10^{-6})$	448	447	447 389
152	$\ln(1+5\cdot10^{-6})$	457	458	457 806
151	$\ln(1-1\cdot10^{-6})$	858	857	857 267
152	$\ln(1+1\cdot10^{-6})$	523	524	523 684

H. S. UHLER, 8 Jan. 1943

On 2 February 1943 Mr. Uhler drew my attention to the fact that five more last-figure errors in Grimpen's 84-place table on p. XXV are suggested by comparison with H. M. Parkhurst's 102-place table (see RMT **86,** p. 20); in the cases of log 23, log 41, log 61 and log 97 there should be unit increases, but in the case of log 83 there should be a unit decrease. I found that Parkhurst and Grimpen were in complete agreement in the cases of log 31, log 43 and log 59, referred to above; hence it is Sharp's terminal digits which seem then to be slightly erroneous. On 3 May 1943 Mr. Uhler reported that he had completely checked both of Grimpen's tables, p. XXIV–XXV, and that the only errors were those in the terminal figure indicated above.—EDITOR.

The correct value of π to 707D was calculated by William Shanks and may be found on p. 1 of the *Anhang* by P. & S., and in G. Peano, *Formulario Mathematico*, 5 ed., v. 5, Turin, 1908, p. 256. Shanks gave the value of π to 607D in his *Contributions of Mathematics, comprising chiefly the Rectification of the Circle* . . . , London, 1853, p. 86–87. That the last 8 digits were incorrect, was shown when he extended his value of π to 707D, giving at the same time arctan (1/5) and arctan (1/239), each to 709D, R. So. London, *Proc.*, v. 21, 1873, p. 319. But there were still errors in the 460–462nd, and in the 513–515th decimal places. These were corrected in the value Shanks gave, *idem*, v. 22, 1874, p. 45. Two new errors here introduced in the 326th and 680th decimal places were easily checked from the arctangent values referred to above, and used by Shanks in computing the value of π. See RMT **95.** A reference may be given to J. P. Ballantine "The best (?) formula for computing π to a thousand places," Amer. Math. Mo., v. 46, 1939, p. 499–501.

R. C. A.

UNPUBLISHED MATHEMATICAL TABLES

We have referred to unpublished mathematical tables (a) of COMRIE in RMT **82;** and (b) of RICHE DE PRONY, of SANG, of PETERS, and of Princeton University, in the first article of this issue.

2[D].—J. R. AIREY, *Sines and Cosines in Radian Arguments*. Ms. in Mr. Comrie's possession.

After the death of Airey in 1937, his calculations and manuscript tables came into my possession. Most of these had, of course, been published, although in many cases, e.g., the Fresnel integral, more decimals (usually within a unit of the last decimal) thus became available.

There is one unpublished table of not inconsiderable value, especially to table-makers. It consists of four neatly written foolscap volumes, with 31 lines to a page, giving sines and cosines to 13 decimals at interval 0·0001 radians up to 0·8 radians, i.e., just past the first octant. From this, with the aid of multiples of $\frac{1}{2}\pi$, any sine or cosine can be found. The table can best be described by illustrating a typical opening with headings supplied.

Left hand page					*Right hand page*			
sin x	Δ′	sin x	Δ′	x	cos x	Δ′	cos x	Δ′
·16721 08424 8		827		·1680	·98592 11602 1		132·	
	9 85912 8		798			1 67260 1		138
30 94337 6		625		1	90 44342 0		994·	
	896 1		067			7358 7		728
40 80233 7		692		2	88 76983 3		266	

The first column gives the sine to 11 decimals, and the second its difference. The third column gives the 11th, 12th and 13th decimals, which can replace the previous 11th decimal, with lowering by 1 of the tenth decimal when the 11th is 0 and this group is 950 or more, as in the case of cos $0^r\cdot1681$. The next column gives the 11th, 12th and 13th decimals of the first difference, which can replace the 11th decimal of the previous difference, as before. The arrangement on the right hand page is similar.

Actually the table was first made to 11D, the number used in Airey's table at interval $0^r\cdot001$ in the Br. Ass. Adv. Sci., *Report* for 1916, and in its *Mathematical Tables*, v. 1, 1931. The decision to extend it to 13D was made later. The 13th decimal has been worked to approximately half a unit, the half unit being indicated by a following central dot.

In 1931 the 11-figure values were compared under the present writer's direction with a 10-figure table (to 1·6 radians) at the same interval that he had made on a Burroughs machine. This enabled 12 errors in Airey's table to be corrected, as well as 10-end-figure errors between 0·8 and 1·2 units in the 10-figure table.

The published tables at interval $0^r\cdot001$ are the Br. Ass. (Airey's) table mentioned above, Van Orstrand's 23-place table in Nat. Acad. Sci., Washington, *Memoirs*, v. 14, part V, 1925, and Hayashi's 12-place (after $x=0\cdot1$) table in *Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen*, Berlin, 1926, which is notoriously inaccurate, and should not be used unless corrected. There is only one table at interval $0^r\cdot0001$, namely the New York W.P.A. *Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments*, 1939, to 9D. All these tables go to $1^r\cdot6$ or 2^r, and so have the slight advantage that multiples of $\frac{1}{2}\pi$ can always be *subtracted from* a large angle.

A table at interval $0^r\cdot001$ is virtually linear to seven decimals, and one at interval $0^r\cdot0001$ to nine. If we need twelve decimals in order to produce accurate 10-figure tables involving sines and cosines, an interval of $0^r\cdot0001$ is desirable in order that the effect of third differences may be negligible. Second differences are, of course, simply the first four decimals of the function, with a negative sign.

There is no doubt that Airey's manuscript could be used for an 80-page 12-decimal table that would be accurate to about 0·6 units of the last decimal—ample for all computing purposes. Where the 13th decimal is 4· (which might be 4 or 5), it would be rounded down, and 5· would be rounded up. A 5 (i.e., something between 4·75 and 5·25) would be treated in accordance with the rule that leaves the last figure even. That Airey had hoped to publish the table (perhaps privately) is shown by a printer's 12-line specimen, in two different types, in one of the volumes. I should be glad to have any expressions of opinion as to the advisability or otherwise of publication, bearing in mind the cost of printing, the existence of the W.P.A. table at the same interval to nine decimals, and the limited demand for such a table.

L. J. C.

3[L].—L. J. Comrie, *The Bessel Functions J_0 and J_1*. Ms. in Mr. Comrie's possession.

I have two bound Burroughs-script tables, one of J_0 and the other of J_1, both being to 12D, for $x=0(0\cdot001)16(0\cdot01)25$. The argument is printed in small red figures, and the function (which has the decimal printed in the correct position) has been ruled into groups of 5, 5 and 2 figures. First and second differences are given, the former being in complementary form when they have the opposite sign to that of the function; the latter are always in direct form. The tables

are printed on foolscap (one side only), with the usual machine spacing of one-sixth of an inch.

These tables were prepared about 1933–34, while I was engaged on producing the Br. Ass. Adv. Sci., *Mathematical Tables*, v. 6, *Bessel Function*, part I, 1936, in which these values appear (with their second differences) to 10D. The part up to $x=15\cdot5$ was formed by subtabulating to tenths the 12-figure table of Ernst Meissel, as given originally in the *Berliner Abhandlungen* for 1888, and reprinted in A. Gray and G. B. Mathews, *A Treatise on Bessel Functions*, London, Macmillan, 1895; 2nd ed. by Gray and T. M. Macrobert, 1922. For the range $x=15\cdot5$ to $x=25$, the sources were a manuscript table lent by H. T. Davis, original calculations based on Meissel's table of $J_n(x)$ for integral values of n and x (in Gray and Mathews), and comparison with Hayashi's *Tafeln der Besselschen, Theta, Kugel- und anderer Funktionen*, Berlin, Springer, 1930. The latter contains 22 errors in this range.

The tables were a stepping stone to the published 10-figure tables mentioned. They are, however, being retained in case these values should ever be required to more decimals.

L. J. C.

MECHANICAL AIDS TO COMPUTATION

2[Z].—S. Lilley, "Mathematical Machines," *Nature*, v. 149, 25 Apr. 1942, p. 462–465.

This is a pleasantly written survey article, beginning with the "rise of modern arithmetic" (based on material in Stevin's decimal arithmetic of 1585, and Napier's logarithms of 1614), then on to discussion of "future trends." The whole concludes with an excellent selected bibliography. Its a good article for the uninformed reader, desiring to get a general idea of achievements up to the present, especially if he reads also about half of the score of sources (1914–40) which have been brought to his attention.

The earliest calculating machines of Pascal, Morland, and Leibniz, discussed in MAC 1, are merely mentioned, as well as several others dating from the eighteenth century. Taken together such machines specializing in the operations of addition, subtraction, multiplication, and division. But each of these machines was a failure, because "industrial revolution" had not caused the development of techniques for producing gears of adequate exactness and durability. Such a revolution took place in Great Britain in the first half of the nineteenth century when problems of power and its transmission, arising from the steam engine, were effectively handled. In this period Charles Babbage[1] (1792–1871), mathematician and scientific mechanician, was the only one appreciating economic trends and attempting to employ mathematics to assist in this work. His remarkable Difference Engine (invented in 1812), and, the incomplete "Analytical Engine" (1833+), of which the Hollerith punch-card machine is the modern counterpart, were products of his genius.

We are told that with the possible exception of the arithmometer (1820) designed and introduced by Charles X. Thomas of Colmar, no successful machine was produced until the 1880's, when the continually growing demand, coupled with the accurate machine tools which were then available caused an extremely rapid development of many efficient machines. The Comptometer (1877) invented by D. E. Felt of Chicago, was the first successful Key-operated machine; and many other successful machines were made about the same time. The German Brunsviga Calculating Machine,[2] based on the invention of a Russian engineer, first appeared in 1892, and the completion of the 20000th machine was celebrated in 1912. During the past fifty years no fundamental change has occurred in the ordinary calculating machine, although a multitude of detailed improvements have increased its speed and efficiency. Especially in the late nineteenth and early twentieth centuries did many new specialized machines appear for rapidly solving particular types of problems. Reference may be made to two of these. The National Accounting Machine[3] is very similar to the one planned by Babbage, and has been extensively used in computing the British *Nautical Almanac*. W. J. Eckert tells us[4] that the first extensive use of the early Hollerith in astronomy was made by L. J. Comrie. He used it for building a table from successive differences, and for adding large numbers of harmonic terms.[5] Modern Hollerith machines seem to be capable

of solving an almost endless variety of mathematical problems. For many of these problems tables have been put on punched cards; see Eckert (*l.c.*) p. 39–42, and under "C. *Mils*" in the introductory article of this issue. The Hollerith system was evolved in the U. S. Census Bureau and was used in 1911 in connection with the British census.

In recent years the mechanical solution of differential equations has become a matter of great importance. As long ago as 1876 Kelvin indicated the type of machine[6] for theoretically solving linear differential equations, but mechanical difficulties prevented him from making a model. These difficulties were not overcome until 1931, when Vannevar Bush, now president of the Carnegie Institution of Washington, built the first Differential Analyzer, at the Massachusetts Institute of Technology.[7]

Problems of wave motion arising in innumerable problems of physics led to the construction of many machines such as Kelvin's Tide Predicting Machine[6] (1876) and various Harmonic Analyzers for finding the coefficients in Fourier expansions of periodic functions. Among harmonic analyzers are those of P. Boucherot (1893) in France; O. Henrici (1894), A. Sharp (1894), and G. U. Yule (1894) in England; A. A. Michelson and S. W. Stratton (1898) at the University of Chicago; O. Mader (1909) in Germany. Wave problems derived from radio and quantum mechanics have inspired the construction in this country of harmonic analyzer and synthesizers such as have been described by F. W. Kranz (1927), H. C. Montgomery (1938), and S. L. Brown (1939).

And finally Mr. Lilley refers to important problems in which two out of many other types of machines[8] are used; one type is for solving sets of simultaneous linear equations, and the other type for deriving the solutions of integral equations.

For engineering readers it may be noted that there is a very popular article by E. W. Crew, "Calculating Machines," in *Engineering*, v. 162, 19 Dec. 1941, p. 438–441. It is stated that this article was reprinted from Institution of Electrical Engineers, *Students' Quart. Jn.*, which is not to be found in the principal libraries of this country.

R. C. A.

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[1] The article misleads in baldly referring to Babbage as "Lucasian professor of mathematics at Cambridge," when he held this title for only the eleven years 1828–39, but delivered no lectures.

[2] For the first published account of the application of a commercial calculating machine to mechanical integration see L. J. Comrie, "On the application of the Brunsviga-Dupla calculating machine to double summation with finite differences," R. Astr. So., *Mo. Notices*, v. 88, 1928, p. 447–459. The Brunsviga-Dupla has since been superseded by the Brunsviga 20, the Burroughs, and the National. See also L. J. Comrie, (a) "The application of the Brunsviga Twin 13Z calculating machine to the Hartmann formula for the reduction of prismatic spectograms," *The Observatory*, v. 60, 1937, p. 70–73; (b) *On the Application of the Brunsviga Twin 13Z Calculating Machine to Artillery Survey*, London, Scientific Computing Service, 1938. This latter describes new and original methods of solving problems arising in surveys where rectangular coordinates are used.

[3] L. J. Comrie, "Inverse interpolation" and "Scientific applications of the National Accounting Machine," R. Statist. So., *Jn.*, v. 99, 1936, suppl., v. 3, p. 87–94, 94–114.

[4] W. J. Eckert, *Punched Card Methods in Scientific Computation*, New York, Columbia University, 1940. ix, 136 p.

[5] L. J. Comrie, (a) "On the construction of tables by interpolation," R. Astr. So., *Mo. Notices*, v. 88, 1928, p. 506–523; (b) "The application of the Hollerith tabulating machine to Brown's Tables of the Moon," R. Astr. So., *Mo. Notices*, v. 92, 1932, p. 694–707; (c) "Application of Hollerith equipment to an agricultural investigation," R. Statistical So., *Jn.*, v. 100, 1937, *Suppl.*, v. 4, p. 210. This last item has a full description of the equipment and the way in which it may be used for forming sums of products. See also Comrie's *The Hollerith and Powers Tabulating Machines*, London, printed for private circulation, 1933, 48 p.; based on lectures, many illustrations.

[6] An account of Kelvin's machines is given in W. Thomson and P. G. Tait, *Treatise on Natural Philosophy*, new ed., v. 1, pt. 1, Cambridge, 1879, app. B′, p. 479–508.

[7] V. Bush, "The differential analyzer. A new machine for solving differential equations," Franklin Inst., *Jn.*, v. 212, 1931, p. 447–488, well documented. The original machine has been greatly extended and improved. The best simple account of the differential analyzer was by D. R. Hartree in "The mechanical integration of differential equations," *Math. Gazatte*, v. 22, 1938, p. 342–363, with five admirable plates, reproductions of photographs. There are also interesting "References" especially to "Applications."

[8] V. Bush, "Recent progress in analysing machines," *Proc. Fourth Intern. Congress for Applied Math., Cambridge, England . . . 1934*, Cambridge, 1935, p. 3–23. Admirable review of progress to 1934.

3[Z].—L. J. COMRIE, "Calculating machines," Appendix III in L. R. Connor, *Statistics in Theory and Practice*, London, Pitman, 1938, p. 349–371.

In these 23 pages, with 14 illustrations of machines, is an admirable and condensed account of the main types of current machines, with some indication of the arithmetical field in which each might most advantageously be used. The machines are discussed under three headings: (a) Adding and listing machines; (b) Calculating machines; (c) Punched card sorting and tabulating machines.

Under the first heading the following machines are mentioned: Full Keyboard (Burroughs, Continental, National, and Victor); Ten-key (Sundstrand, Remington, Burroughs Typewriter Accounting); Key-driven (Burroughs Calculator, Felt and Tarrant Comptometer, and the Plus); Typewriter Accounting (Elliott Fisher, Mercedes-Euklid, Remington, Smith Premier, and Underwood); Multi-register (Burroughs Typewriter Bookeeping, Sunstrand, National Cash Register Analysis); National Accounting Machine.

The term calculating machine is properly applied to a machine which caters primarily for multiplication and division. Examples listed of such machines are the hand-operated Brunsviga; the electrically operated Mercedes-Euklid (German), Madas (Swiss), Archimedes, Marchant (American), Facit (Swedish) and Monroe (American); and the direct-multiplication machines, Millionaire, certain models of the Burroughs Bookkeeping machine, and the Hollerith multiplying punch.

Under the third heading are listed, Hollerith Sorter, Hollerith Electric Rolling Total Tabulator, and Hollerith Multiplying Punch.

The whole concludes with an annotated reference list, some use of which was made in MAC 2.

Neither Mr. Lilley nor Mr. Comrie refer to the electric Fridén Automatic Calculator made in California; it has been on the market since 1934. A small condenser in the machine prevents the lights of the room being affected by its use; it is fast; and it is completely automatic.

4[Z].—(i) C. E. SHANNON, "Mathematical theory of the differential analyzer," *Jn. Math. Phys.* (M. I. T), v. 20, 1941, p. 337–354.
(ii) H. S. W. MASSEY, J. WYLIE, R. A. BUCKINGHAM, R. SULLIVAN, "A small scale differential analyser—its construction and operation," Irish Acad., *Proc.*, v. 45A, no. 1, Oct. 1938, 21 p.+5 plates.

The possibilities of applying modern technical developments in machines and electronics to both elementary and advanced calculations are very great. To date these possibilities remain relatively unexplored. An exception is the development by V. Bush in 1931 of the differential analyzer,—a mathematical tool which may well be regarded as the most extraordinary of our time. To a mathematician the most remarkable feature of the machine is its ability to solve non-linear differential equations (or systems) as readily as linear ones. Another interesting feature is that the machine can introduce functional relations automatically, provided these functions are themselves solutions of algebraic differential equations with constant coefficients. The main limitations of the differential analyzer are: (a) Only the one-point boundary problem can be solved directly. The two-point boundary problem must be solved by trial, by successively adjusting the initial conditions at one point until the solution given by the machine satisfies the given conditions at the second point. This objection would be overcome if the analyzer could be improved by increasing its rate of speed (or by simplifying the machine to the point where one could have several such analyzers to operate simultaneously). (b) The machine fails near a singular point where any of the variables or derivatives is unbounded.

In paper (i) the author points out a third limitation of a more technical nature. The analyzer consists of a number of units called integrators, adders, multipliers, etc. which are interconnected in a manner determined by the differential equation being solved. These units are unidirectional in operation, that is, one part of the unit will accept information from other units of the machine (input side) and another part will dispense information (output side), but the input and output sides cannot be interchanged. For instance, the integrators cannot be run as differentiators by

interchanging the roles of the input and output sides; the machine can integrate an empirical curve but cannot differentiate one.

This limitation is not a serious one, for the author shows that any differential equation of the form $f(x, y, y', \cdots, y^{(n)})=0$ can be solved provided only that the function f is not hypertranscendental in character, that is, provided f considered as a function of any of its arguments satisfies a differential equation of the form $P(x, y, y', \cdots, y^{(m)})=0$ where P is a polynomial. The solution can be accomplished with a finite, but sufficiently large, number of units, and without any information supplied to the machine by the operator beyond the connections between the units demanded by the particular problem. Furthermore, even though f is hypertranscendental, it will still be possible to approximate the solution in the sense that a function y can be obtained which makes the left member of the given equation uniformly less than a given positive number ϵ. Or, the operator can supply the hypertranscendental function to the machine.

That differential analyzers are not in general use is due to their prohibitive cost, being of the order of many thousands of dollars for the larger machines. In paper (ii) there is described in detail the construction of a small scale analyzer which, while relatively inexpensive, is claimed to have an average accuracy of one half percent. The authors have reduced the cost of the machine by limiting the number of integrators to four, by eliminating certain refinements such as backlash compensators, automatic speed controls, etc., and by using standard parts for the construction. The resulting cost of materials is given as about £50, but the time required to assemble the parts must have been considerable. One wonders if the construction of machines following the authors design might not be brought within the range of the abilities and interests of groups of amateur hobbyists if fostered by mathematicians after the fashion set by astronomers in encouraging the construction and use of reflecting telescopes. That the completed machines are in the nature of glorified toys must have been in Hartree's mind when he in 1935 succeeded in making a demonstration model out of toy meccano parts.[1]

P. W. KETCHUM

NOTES

4. GIFFORD AND C. G. S. TABLES.—In RMT 77 the improvements in the Table of *Natural Sines and Cosines* published by the Coast and Geodetic Survey, as compared with Emma Gifford's volume, on which it was mainly based, were not made sufficiently clear. The Gifford volume is defective in that
(a) The arrangement of sines throughout the quadrant so that sines and cosines of a given angle have to be sought in different places.
(b) Consecutive values are in rows rather than in columns.
(c) The number of errors of more than a unit in the last decimal place is very large. The C. G. S. table is a notable improvement by virtue of
(a′–b′) Its semi-quadrantal arrangement, in columns, of the sines on one page and the cosines on the opposite page. (That an arrangement with sines and cosines on the same page would have been still better can hardly be gainsaid.)
(c′) The number of errors of more than a unit in the last decimal place being almost negligible. One may add (d) cost of Gifford 40 shillings, (d′) cost of C. G. S. 1.75 dollars.

R. C. A.

5. COAST AND GEODETIC SURVEY VERSUS PETERS.—In MTE 1, p. 25, lines 6–12 from the bottom, six entries in C. G. S., *Natural Sines and Cosines* (RMT 77) were called into question by quoting the corresponding results in Peters' *Eight-figure Table of the Trigonometrical Functions* (RMT 78). On

[1] D. R. Hartree and A. Porter, "The construction and operation of a model differential analyser," Manchester Lit. and Phil. So., *Mem. and Proc.*, v. 79, 1935, p. 51–71, +2 plates.—EDITOR.

5 March, 1943, G. M. CLEMENCE, of the U. S. Naval Observatory, wrote as follows: "As a matter of academic interest I have taken these six sines out of Andoyer's 15-place tables, obtaining the following results for the decimals from the fifth to the twelfth inclusive:

17°15′19″	2954	4997	68°49′06″	3946	5041
33 20 15	6973	5006	71 09 45	3813	4969
61 01 41	5699	5037	74 02 28	5922	4959

It appears that Peters is correct in every instance, the errors in the Survey tables being in units of the twelfth decimal, 5003, 5006, 5037, 5041, 5031, 5041. These errors are hardly important in computation, the largest being less than 1 percent greater than the unavoidable error in using an 8-place table."

6. THE EIGHT-FIGURE TABLE OF J. T. PETERS AND L. J. COMRIE.—In RMT **78** we referred to *Eight-figure Table of the Trigonometrical Functions for Every Sexagesimal Second of the Quadrant*, produced by the British War Office in 1939 and 1940 from the German original, and pointed out that this volume was unfortunately not available even to British scientists. L. J. Comrie has set forth the merits of this great table as follows:

"(1) All four functions of any angle are on the same line in four adjacent columns.

(2) Consecutive values are in columns, not in lines.

(3) The printing is perfect.

(4) The table is free from error. Not one of the discrepancies between Peters and Gifford's *Sines* and *Tangents* (which have upwards of 700 errors) was traced to Peters. The mechanical printing of the printer's copy, and Peters' known standard of proof reading (including the reading against an independently prepared copy) are such as to inspire the utmost confidence."

Since another table of Peters (see RMT **79**) has been "published and distributed in the public interest by authority of the Alien Property Custodian," we urge most strongly that at an early date this larger work be also made available to scientific workers.

7. SMITHSONIAN TABLES.—It seems desirable to place on record some notes concerning these five volumes of Washington tables, of uniform format, about 16×24 cm. Among the early publications of the Smithsonian Institution was an important volume of ARNOLD HENRY GUYOT (1807–1884), *Tables, Meteorological and Physical*, 1852; second edition revised and enlarged, 1858; fourth edition, 1884, xxv, 747 p. Thereafter the work was divided into the following three different publications:

I. *Meteorological Tables*, 1893. This is now Publication 3116, fifth revised edition (corrected to 1931), reprinted with corrections 1939, xiii, 282 p. The 116 tables include those which are: Thermometrical; Hygrometrical; Geodetical; Involving Conversions of Linear Measures; Measures of Time and Angle, and Measures of Weight; For Determining Heights and Conversions involving Geopotential.

II. *Geographical Tables*, 1894. Publication 854, third edition, second reprint, 1929, cv, 182 p. This last edition was prepared by R. S. WOODWARD. The introductory pages set forth useful formulae, etc., in Mathematics, Geodesy, Astronomy, and Theory of errors. Then follow 42 tables which include, for example, Natural sines, cosines, tangents, cotangents, to 4D;

Coordinates for projection of maps, in various scales; Areas of quadrilaterals of the earth's surface of varying extents in latitude and longitude.

III. *Physical Tables*, 1897. Publication 8171, eighth revised edition by F. E. Fowle (1932), first reprint, 1934, liv, 686 p. The introductory pages deal mainly with various formulae and units. The 874 tables include the following: Mathematical Tables; Tables of Mechanical Properties, Acoustics, Aerodynamics, Viscosity, Radiation, The Eye and Radiation, Electromotive Forces, Electrolysis, Atomic Structure, Radioactivity, Meteorology, Geodesy, Geophysics, Terrestrial Magnetism, Astronomy, Nebulae. The Mathematical Tables included exponential functions and their logarithms, diffusion integral, exponential integral, gamma function, zonal spherical harmonics, cylindrical harmonics, elliptic integrals.

To these three volumes was added

IV.—Publication no. 1871. *Smithsonian Mathematical Tables, Hyperbolic Functions*, 1909, of which the fifth reprint was published in 1942, lii, 321 p. Corrections of errors were made in each of the reprints. The Secretary of the Smithsonian wrote in part as follows: "Hyperbolic functions are extremely useful in every branch of pure physics, and in the applications of physics whether to observational and experimental sciences or to technology. Thus whenever an entity (such as light, velocity, electricity, or radioactivity) is subject to gradual extinction or absorption, the decay is represented by some form of Hyperbolic Functions. Mercator's projection is likewise computed by Hyperbolic Functions. Hence geological deformations invariably lead to such expression, and it is for that reason that Messrs. Becker and Van Orstrand, who are in charge of the physical work of the United States Geological Survey have been led to prepare this volume." Compare RMT **89**.

The final volume in this series is

V.—Publication 2672. *Smithsonian Mathematical Formulae and Tables of Elliptic Functions, Mathematical Formulae prepared by* Edwin P. Adams . . . *Tables of Elliptic Functions prepared under the Direction of* George Greenhill, by R. L. Hippisley, 1922; first reprint, 1939, viii, 314 p. In this reprint a few errors have been corrected. It was not originally intended to include any tables of functions in this volume, but merely to give references to such tables. But an exception was made in favor of the Greenhill-Hippisley table calculated on a new plan (in 1922) and not otherwise available. Greenhill wrote the introduction to these tables (p. 243–258). F. R. Moulton is the author of section X (p. 220–242) on "Numerical solution of differential equations." Sections VIII–IX are devoted to formulae and bibliography of "Differential equations." Other topics treated are Algebra, Geometry, Trigonometry, Vector analysis, Curvilinear coördinates, Infinite series, and Special applications of analysis.

QUERIES

2. Scarce Mathematical Tables.—In what libraries of the world, public or private, may the following books be found:

A. Achille Brocot, *Calcul des Rouages par Approximation, Nouvelle Méthode*. Paris, l'auteur, 1862. 97 p.

B. Achille Brocot, *Table de Conversion en Décimale des fonctions ordinaire a l'Usage du Calcul des Rouages par approximation. Méthode Nouvelle.* Paris, P. Dupont, 1862, 51 p.

C. Berechnung der Räderübersetzung. Herausgegeben von dem Verein "Hütte." Bearbeitet nach Calcul des Rouages par Approximation, Nouvelle Méthode par Achille Brocot. Berlin, 1871, xvi, 52 p.

D. Idem, second ed., Berlin, 1879, 67 p.

E. [Henry Goodwyn], *A Table of Circles arising from the Division of a Unit or any other Whole Number, by all the Integers from* 1 *to* 1024, *being all the pure Decimal Quotients that can arise from this source.* London, 1823 v, 118 p. Published anonymously.

We have already noted (RMT **87**, p. 21) that *A* and *B* were formerly in the Bibliothèque Nationale, Paris.

R. C. A.

QUERIES—REPLIES

2. Tables to many places of Decimals (Q1; QR1).—The functions which occur in the solution of the problems of applied mathematics are of numerous types but many of them are associated with the differential equation of the hypergeometric function and its generalizations. Now it happens that the logarithmic case of this equation is of frequent occurrence and the desired solution consists of two parts one of which is multiplied by a logarithm. The Legendre function of the second kind

$$Q_n(z)=\tfrac{1}{2}\int_{-1}^{+1}dtP_n(t)/(z-t)$$

is a type of such a function and in computations it is generally convenient to use the recurrence relation

$$(n+1)Q_{n+1}(z)+nQ_{n-1}(z)=(2n+1)zQ_n(z)$$

which is satisfied by each of the two terms of which $Q_n(z)$ is composed. Now when z exceeds unity these two terms are very nearly equal, but of opposite sign: consequently the desired value of $Q_n(z)$ is the difference of two quantities which must be known very accurately. The first two functions are

$$Q_0(z)=\tfrac{1}{2}\ln[(z+1)/(z-1)],\ Q_1(z)=zQ_0(z)-1$$

and so the logarithm giving the value of $Q_0(z)$ must be found to a large number of places of decimals. In some calculations that were made in connection with a hydrodynamical problem use was made of the values given by J. C. Adams in his note on the value of Euler's constant (R. So. London, *Proc.*, v. 27, 1878, p. 88–94, *Scientific Papers*, v. 1, Cambridge, 1896). The short tables of $Q_n(z)$, published in *Messenger of Math.*, v. 52, 1923, p. 71–78, were actually calculated with the aid of the recurrence relation and it was found that the difference between nQ_{n-1} and $(2n+1)zQ_n$ was generally quite small. Further calculations have been made by this method for many integral values of z so as to have about 30D in the value of $Q_n(z)$ many of the figures being zeros. Such accuracy may not ordinarily be needed but expansions in series of $Q_n(z)$ are useful in the solution of problems in hydrodynamics and sometimes the coefficients are large. It should be mentioned that a resolution of a function into two terms one of which has a logarithmic factor occurs naturally in the evaluation of certain integrals which occur in potential theory. When, for instance, the integrand is of form $P_n(t)/(z-t)$, the subtraction of an integral whose integrand is $P_n(z)/(z-t)$ gives an integral whose integrand is a polynomial. Sometimes the integral-logarithm takes the place of the logarithm in the first

of two terms representing an integral. This is the case, for instance, when an integral used by T. H. Havelock is resolved into two parts each of which satisfies a certain recurrence relation. The integral in question represents a logarithmic case of the confluent hypergeometric function and occurs in the paper, "The method of images in some problems of surface waves," R. So. London, *Proc.*, A iv. 115, 1927, p. 268–280. Many figures may be needed, then, in tables of the integral-logarithm and in the value of Euler's constant which occurs in many expressions for this function. Tables to many places of decimals are needed occasionally for the solution of transcendental equations. In his paper "Comparaison de la méthode d'approximation de Newton à celle dite des parties proportionnelles," *Nouv. Annales d. Math.*, s. 2, v. 18, 1879, p. 218–231, L. Maleyx calculates the root of

$$e^x - x^e = \tfrac{1}{2}\pi$$

which lies between 1 and e by two different methods and finds that with 6 substitutions the method of proportional parts is more accurate than the Newton-Raphson method. He says that his calculations were made with the aid of the excellent tables of Fédor Thoman, *Tables de Logarithmes à 27 Décimales pour les Calculs de Précision*, Paris, 1867

H. B.

CORRIGENDA

Omit last four lines page 25.

See Note 2 of the article "Tables and trigonometric functions in non-sexagesimal arguments" in this issue of *MTAC*, p. 44.

P. 20, l. 4 from bottom, *for* "II, III, IX," *read* "II, III, IX, XIII."

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CLASSIFICATION OF TABLES, AND SUBCOMMITTEES

A. Arithmetical Tables. Mathematical Constants
B. Tables of Powers
C. Logarithms
D. Circular Functions
E. Hyperbolic and Exponential Functions
  Professor DAVIS, *chairman*, Professor ELDER
  Professor KETCHUM, Doctor LOWAN

F. Theory of Numbers
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G. Higher Algebra
  Professor LEHMER

H. Tables for the Numerical Solution of Equations
J. Summation of Series

I. Tables connected with Finite Differences. Interpolation
K. Statistical Tables
  Professor WILKS, *chairman*, Professor COCHRAN, Professor CRAIG
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L. Higher Mathematical Functions
M. Integral Tables
  Professor BATEMAN

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O. Actuarial Tables
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EDITORIAL NOTICES

The addresses of all contributors to each issue of *MTAC* are given in that issue, those of the Committee being on cover 2. The use of initials only indicates a member of the executive committee.

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